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A Problem of Hartmanis on Generalized Partitions

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Abstract. In 1959, Turing Award winner Juris Hartmanis studied lattices of subspaces of generalized partitions ("partitions of type n"; "geometries" if $n = 2$). Hartmanis states it is "an unsolved problem whether there are any incomplete lattice homomorphisms in" lattices of subspaces of geometries. We give a positive answer to this question.

Keywords: Generalized partitions, lattice theory

1 Introduction

Hartmanis analyzed an abstraction of the concept of partition and called these new objects *generalized partitions*. They can be used to define abstract geometries amongst other things.

Partitions and their generalizations abound in computer science. They appear in fields as diverse as data analysis, search algorithms, computer graphics, and more. Even though the objects that Harmanis studies in his paper (Hartmanis, 1959) are quite "old", their simplicity and abstraction makes them interesting for present, and (we are confident) future computer science.

It turns out that generalized partitions have subspaces with certain closure properties which we define below. In many categories (such as the category of groups or vector spaces over $\mathbb{R}$), the class of subobjects can be ordered in a natural way using the subset relation $\subseteq$ and are closed under arbitrary intersections. They form an algebraic object called a *lattice*, a concept that we look at more closely in the next section.

Hartmanis asked a question about lattices of subspaces of generalized geometries, namely whether they allow for certain lattice homomorphisms. This article gives a positive answer to this question.

2 Notions of lattice theory

The book (Davey et al, 2002) offers an excellent introduction into lattice theory. Nevertheless, we want to define the most important concepts of lattice theory that we use later on.

A *partially ordered set* (or *poset* for short) is a set X with a binary relation $\leq$ that is reflexive, transitive, and anti-symmetric (i.e., $x, y \in X$ with $x \leq y$ and $y \leq x$ implies $x = y$). Often, a poset is denoted by $(X, \leq)$. A subset $D \subseteq X$ is called a *down-set* if it is "closed under going down", that is $d \in D, x \in X, x \leq d$ jointly imply $x \in D$. A special case of a down-set is the set

$$\downarrow x = \{y \in X : y \leq x\}$$

for $x \in X$. Down-sets of this form are called *principal*. If $S \subseteq X$ we say S has a *smallest element* $s_0 \in S$ if $s_0 \leq s$ for all $s \in S$. Note that anti-symmetry of $\leq$ implies that a smallest element is unique (if it exists at all!). Similarly, we define a largest element. Moreover, we set

$$S^u = \{x \in X : x \geq s \text{ for all } s \in S\}$$

to be the *set of upper bounds* of S. The set of lower bounds S^ℓ is defined analogously.

We say that a subset $S \subseteq X$ of a poset $(X, \leq)$ has an *infimum* or *largest lower bound* if

1. $S^\ell \neq \emptyset$, and
2. S^ℓ has a largest element.

Again, an infimum (if it exists) is unique by anti-symmetry of the ordering relation, and it is denoted by inf(S) or $\bigwedge_X S$. The dual notion (everything taken "upside down" in the poset) is called *supremum* and is denoted by sup(S) or $\bigvee_X S$. The infimum of the empty set is defined to be the largest element of X if it has one, and the supremum is the smallest element of X.

A poset $(X, \leq)$ in which infima and suprema exist for all $S \subseteq X$ is called a *complete lattice*. A *lattice* has suprema and infima for finite non-empty subsets. If $(X, \leq)$ is a poset and $x, y \in X$ we use the following notation

$$x \vee y := \bigvee_X \{x, y\},$$

and $x \wedge y$ is defined analogously. To emphasize the binary operations $\vee, \wedge$, a lattice $(L, \leq)$ is sometimes written as $(L, \vee, \wedge)$. A lattice is *distributive* if for all $x, y, z \in L$ we have

$$x \wedge (y \vee z) = (x \vee y) \wedge (x \vee z).$$

An *ideal* of a lattice is a subset $I \subseteq L$ that is a down-set of $(L, \leq)$ and is closed under $\vee$, that is

$$a, b \in I \Longrightarrow a \vee b \in I.$$

The dual notion is called a *filter*. An ideal $I \subseteq L$ is *prime* if $L \setminus I$ is a filter. In other words you cannot "get into I from outside" by applying $\wedge$. This compares well to the situation of ideals in rings: an ideal in a ring is closed under $+$ and it is prime if you cannot get inside the ideal using multiplication with elements outside the ideal.

In distributive lattices, Zorn's Lemma implies an important tool:

Prime Ideal Theorem (PIT): *If L is a distributive lattice, and $I \subseteq L$ is an ideal, and $F \subseteq L$ a filter with $I \cap F = \emptyset$, then there is a prime ideal $P \subseteq L$ such that*

1. *$I \subseteq P$, and*
2. *$P \cap F = \emptyset$.*

Theorem 10.18 in (Davey et al, 2002) offers a detailled proof. (PIT) cannot be proved within ZF, standard proofs use the Axiom of Choice. Interestingly, (PIT) is strictly weaker than the axiom of choice. These subtleties are also explored in the book by Davey and Priestley.

3 Generalized partitions

A *partition of type* n for $n \geq 1$ on a set S (consisting of at least n elements) is a set $\mathfrak{P} \subseteq \mathcal{P}(S)$ such that

1. all members of $\mathfrak{P}$ have at least n elements, and
2. any n elements of S are contained in exactly one member of $\mathfrak{P}$.

Partitions of type 1 are the "traditional" partitions.

A partition of type 2 is referred to as a *geometry*, and its elements are called *lines*.

Definition 1. *If $\mathfrak{G}$ is a geometry on a set S, a set $T \subseteq S$ is said to be a* subspace of S with respect to $\mathfrak{G}$ *if it is "closed under lines," that is, for any distinct $x, y \in T$, for the (unique) element $g \in \mathfrak{G}$ that satisfies $\{x, y\} \subseteq g$ we have $g \subseteq T$.*

We denote the collection of subspaces of S with respect to the geometry $\mathfrak{G}$ by $\mathrm{Sub}(S, \mathfrak{G})$.

If $A \subseteq \mathrm{Sub}(S, \mathfrak{G})$ it is easy to see that $\bigcap A \in \mathrm{Sub}(S, \mathfrak{G})$, therefore $\mathrm{Sub}(S, \mathfrak{G})$ is a complete lattice with respect to set inclusion.

4 Incomplete lattice homomorphisms

Let K, L be complete lattices. If $f : K \to L$ is order-preserving and $S \subseteq L$ we have

$$f(\bigvee_K S) \geq f(s) \text{ for all } s \in S,$$

which implies
$$\bigvee_L f(S) \leq f(\bigvee_K S).$$

A lattice homomorphism $f : K \to L$ is said to be *join-incomplete* if there is $S \subseteq K$ non-empty such that $\bigvee_L f(S) < f(\bigvee_K S)$. (Dually, we define *meet-incompleteness*.) We say f is *incomplete* if it is join-incomplete, meet-incomplete, or both.

The next lemmas deal with incomplete lattice homomorphisms in the context of infinite complete lattices.

Lemma 1. *Let L be an infinite complete lattice (not necessarily distributive) with bottom element 0 and top element 1. Suppose $P \subseteq L$ is a non-principal prime ideal (i.e., $\bigvee P \notin P$). Then there is a join-incomplete lattice homomorphism $f : L \to L$ preserving 0 and 1.*

Proof. Let $f : L \to L$ be 0 on P and 1 on $L \setminus P$. Since P is a prime ideal, $L \setminus P$ is a filter, which implies that f is a lattice homomorphism. It is (join-)incomplete, because $\bigvee P \notin P$ implies $\bigvee f(P) = 0 \neq 1 = f(\bigvee P)$.

Of course, there is a dual version of Lemma 1 about filters instead of ideals.

Lemma 2. *If L is infinite, complete, and distributive, then it contains either a non-principal prime ideal or a non-principal prime filter.*

Proof. Any infinite distributive lattice contains at least a non-principal ideal or a non-principal filter. We may assume that J is a non-principal ideal, so that $j^* = \bigvee J \notin J$. Let $G = \{y \in L : y \geq j^*\}$ be the principal filter generated by j^*. As $J \cap G = \emptyset$ we can use the Prime Ideal Theorem and get a prime ideal P such that $J \subseteq P$ and $P \cap G = \emptyset$, which implies $j^* \notin P$.

Next we show that P is not principal: if we had $p^* := \bigvee P \in P$ then $J \subseteq P$ would imply $p^* \geq j^* = \bigvee J$ and $j^* \in P$ because P is a down-set. Therefore P is a non-principal prime ideal.

Proposition 1. *Let L be an infinite complete and distributive lattice with bottom element 0 and top element 1. Then there is an incomplete lattice homomorphism $f : L \to L$ respecting 0 and 1.*

Proof. Combine Lemmas 1 and 2.

5 Construction of an example

Turing Award winner Juris Hartmanis' problem is on p. 106 of his paper (Hartmanis, 1959):

> So far we have characterized the complete homomorphisms of the lattices of subspaces of geometries. It remains an unsolved problem whether there are any incomplete homomorphisms in these lattices and if so how can these geometries be characterized.

In this section we tackle the first part of Hartmanis' problem.

It asks whether there is a set S and geometry $\mathfrak{G}$ on S and an **incomplete** lattice homomorphism

$$f : \mathrm{Sub}(S, \mathfrak{G}) \to \mathrm{Sub}(S, \mathfrak{G}).$$

Let $S = \omega$ and set $\mathfrak{G} = \big\{\{m, n\} : m, n \in \omega \wedge m \neq n\big\}$.

It is easy to see that $\mathrm{Sub}(\omega, \mathfrak{G}) = \mathcal{P}(\omega)$.

Since $\mathcal{P}(\omega)$ is distributive, Proposition 1 shows that it allows an incomplete lattice endomorphism.

In fact, we can give a more constructive way of providing an incomplete lattice homomorphism $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$. Let $K \subseteq \mathcal{P}(\omega)$ denote the set of finite subsets of ω and let M denote any maximal ideal containing K (this uses Zorn's Lemma). Let f send every member of M to $\emptyset \in \mathcal{P}(\omega)$ and every member of $\mathcal{P}(\omega) \setminus M$ to $\omega \in \mathcal{P}(\omega)$. Then f is an incomplete lattice homomorphism.

What remains open is to have a characterization of the geometries such that the complete lattice of subspaces allows incomplete endomorphisms.

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