Baltic J. Modern Computing, Vol. 6 (2018), No. 2, 96-106
https://doi.org/10.22364/bjmc.2018.6.2.01

Tetrads and their Counting

Juris ČERŅENOKS, Andrejs CIBULIS

Department of Mathematics, University of Latvia

juris2x@inbox.lv, cibulis@lanet.lv

Abstract. There is no known formula yielding the number of distinct polyominoes of the given number of squares. The same can be said about tetrads made of polyominoes or another polyforms. All *n*-ominoes up to n = 17 have been checked, the number of tetrads and the number of full tetrads have been determined. Some open problems have been proposed.

Keywords: Computer-assisted proofs, polyominoes, tetrads, tritads.

1. Introduction

A *tetrad* is a plane figure made of four congruent shapes in such a way that every pair of shapes has a border arc in common. We shall call this shape a generating element of a tetrad or more briefly – *g-element*. In accordance with the famous populariser of mathematics Martin Gardner (Gardner, 1989) the name 'tetrad' firstly was used by M. R. W. Buckley in the Journal of Recreational Mathematics (1975, Vol. 8.). Already DeMorgan (1806 – 1871) had noticed that no more than four regions on the plane can all be in mutual contact with each other. Here we shall study tetrads that are made of *polyominoes*, i.e. plane shapes formed by joining unit squares edge to edge. If a polyomino consists of exactly n unit squares it is called an *n-omino*.

The tetrads have connections with the famous four-colour problem, which since its formulation in 1852 remained unsolved for a long time, until the mathematicians Kenneth Appel and Wolfgang Hacken (University of Illinois, USA) came up (in 1976) with the announcement that they finally had proved the four-colour-map conjecture. Large-size publications appeared in the next year (Appel, Haken, 1977). For more information on this problem and its history, see e.g. (Benjamin, Chartrand, Zhang, 2015; Dowek, 2015). The solution of this problem was obtained in an unconventional way, by reducing the problem to checking many but the finite number of planar graphs by using a computer. The four-colour problem in mathematics history is considered to be the first major problem to be verified using a computer programme. It is not a coincidence that the very notion tetrad appeared almost at the same time when this famous problem was solved. There have been and will be many discussions on the nature of such proofs, whether they are recognizable as correct admissible proofs in mathematics. It is important to note here that computer-assisted proofs play an increasingly important role in modern mathematics, cf., for example (Dowek, 2015).

With purpose-built computer programmes, all n-ominoes (up to n = 17) have been tested, and it was determined how many of them are those that form the tetrads. We paid a special attention to such computer-assisted proofs that give the opportunity to obtain the proof in the classical sense, if possible, a beautiful proof. As a good example of this type of a computer-assisted proof we mention the proof of the existence of full tetrads, see Theorem 1.

So far, there has not been extensive, systematic study of tetrads. One can find several contributions on polyomino tetrads in (Cibulis, Čerņenoks 2012; 2013) and on the website created by American computer scientist George Sicherman (WEB, a).

2. Generating tetrads of a polyomino

Probably the simplest polyomino tetrad is the one shown in Figure 1. This tetrad is made of 9-ominoes. The smallest polyomino tetrads are made from 8-ominoes or octominoes. Moreover, there are exactly 8 octominoes (out of 369) useful for forming tetrads, see Figure 2. None of these tetrads is full (holeless).

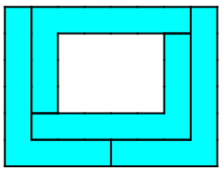

Figure 1. Rectangular tetrad of 9-ominoes.

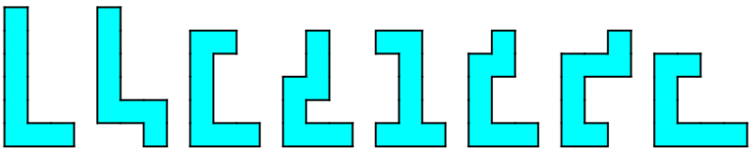

Figure 2. Octominoes useful for tetrads.

There is no known formula yielding the number of distinct polyominoes of the given number of squares. All polyominoes containing up to 24 squares have been enumerated in the article (Redelmeier, 1989). He mentions that it was done by using a faster polyomino enumeration method than previous ones, and spending ten months of computer time. Now the enumeration has been done up to 45-ominoes, see the sequence A000105 in The On-line Encyclopedia of Integer Sequences (WEB, b).

To create and to investigate tetrads the full list of n-ominoes was firstly generated by using the Redelmeier's method. Then each n-omino was analysed separately.

1) The first copy of the given polyomino P is fixed in the first quadrant. The next copy of P is added to the first in all possible ways and they all are stored in array M (cells of polyominoes are coded by complex numbers). If a polyomino is not symmetric, it has 8 different positions.

2) Further, a triple cycle is performed over all elements of M. If every two polyominoes out of three $M(i)$, $M(j)$, $M(k)$, $i < j < k$, do not overlap and have a common boundary, then the tetrad $[P, M(i), M(j), M(k)]$ is found. Of course, it is verified that such a tetrad from P has not been generated already before (for example, by rotation or by a mirror image).

3) The tetrads found are saved and the algorithm continues with the next n-omino.

The results obtained using the computer are summarised in Table 1.

Table 1. Polyomino tetrads.

n	**The number of *n*-ominoes**	**The number of useful *n*-ominoes**	**The number of tetrads**	**The number of full tetrads**
8	369	8	14	0
9	1285	42	83	0
10	4655	187	341	0
11	17073	739	1388	1
12	63600	2871	5648	10
13	238591	11300	22688	1
14	901971	44440	90243	11
15	3426576	172984	352243	36
16	13079255	670107	1373595	66
17	50107909	2599478	5384209	70

Comparing the data of the second and third columns we can hypothesise that the ratio between the number of useful n-ominoes and the number of all n-ominoes is growing. Let us mention that computer search for tetrads made from 17-ominoes took approximately 30 hours (only one thread used). Calculations were performed on processor Intel® Core™ i7-3820 @ 3.60 GHz. Programme was implemented in Pascal.

Remark. This algorithm is not difficult to generalise if polyominoes are replaced by another polyforms.

3. Full tetrads

Generating of full tetrads, even from small polyominoes, is not an easy task. For example, the checking of all 11-ominoes (their total number is 17073) may take several hours of computer time. The smallest full polyomino tetrad uses 11-ominoes. Surprisingly that there is only one 11-omino and only one 13-omino (out of 238591) forming a full tetrad, see Figure 3. Notice that this unique 11-omino serves also as the base to generate infinitely many full tetrads. Moreover, these polyominoes (g-elements) consist of the odd number of squares and form full tetrads without being reflected.

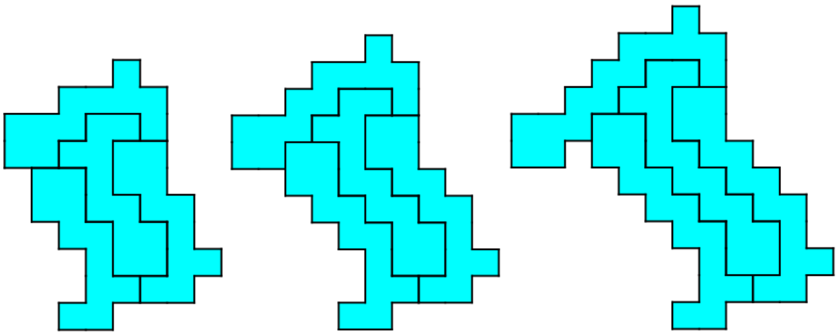

Figure 3. Full tetrads generated by n-ominoes with an odd n.

There are also infinitely many polyominoes consisting of the even number of squares and generating full tetrads without being reflected, see Figures 4 – 5. The smallest n-gon generating a full polyomino tetrad is an octagon, see Figure 4. Is there a symmetric full tetrad? The smallest n, giving a positive answer to this question, is n = 14, see Figure 6. Moreover, there is only one such 14-omino (out of 901971).

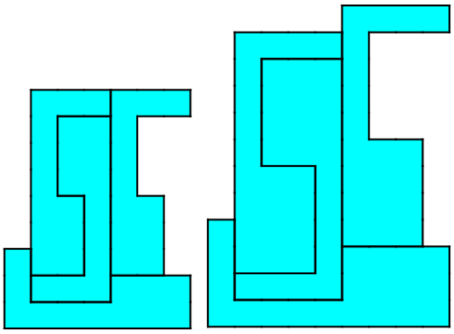

Figure 4. Full tetrads composed of octagons without being reflected.

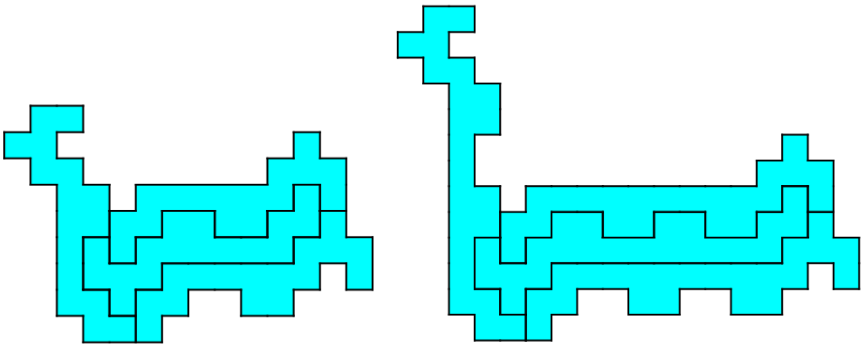

Figure 5. Tetrads composed of $(16 + 6k)$-ominoes without being reflected.

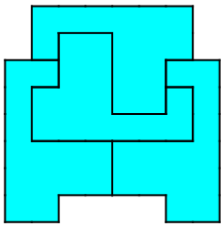

Figure 6. The symmetrical tetrad composed of 14-ominoes.

Theorem 1. For every $n \geq 11$ there is a full polyomino tetrad made of *n*-omino.

The very idea of this theorem is simple. It is necessary to find such a tetrad that can be extended to the next one. It is easily to propose such an idea, but it is hard to find such a tetrad. Let us emphasise that the key how to prove this theorem was found looking through the huge set of tetrads generated by the computer. We managed to find (by a computer) the necessary tetrad made from 17-omino.

Proof. The base tetrad the g-element of which is the 17-omino is shown in Figure 7. Afterwards such a tetrad has been found we can present the proof of the classical type. This tetrad (or its g-element) can be stretched out horizontally, more precisely; we can add any number of unit squares in the indicated places in Figure 8.

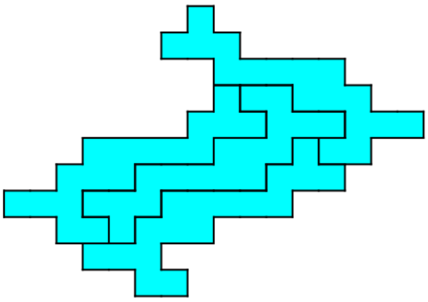

Figure 7. The base tetrad.

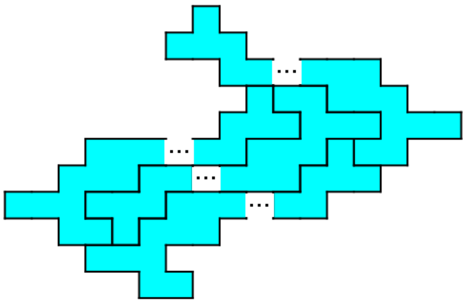

Figure 8. Extension of the tetrad.

Full tetrads for the remaining values n = 11, 12, 13, 14, 15 and 16, are shown in Figures 3 – 6, respectively. The theorem has been proved.

Remark. The possibility is not excluded that there is also a smaller base tetrad in the polyomino class.

4. Tetrads with a prescribed hole

The study of tetrad arrays shows a great variety of holes and suggests the question: what polyominoes can represent the hole in single hole tetrads? The answer is given by such a theorem.

Theorem 2. For each full polyomino *P*, there is a tetrad with the single hole being *P*.

Proof. This theorem has an unexpectedly short proof. Let us fix the lowest square of the polyomino *P*. If the lowest square is not unique we choose the first square on the left and colour it in black, see Figure 9 as an example. Then we place the polyomino *P* on the sufficiently large g-element shown in Figure 10 so that the black squares of both polyominoes coincide with each other. The required tetrad with the hole *P* is shown in Figure 11. The theorem has been proved.

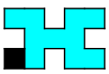

Figure 9. Polyomino P.

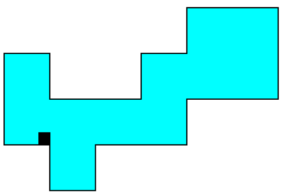

Figure 10. g-element.

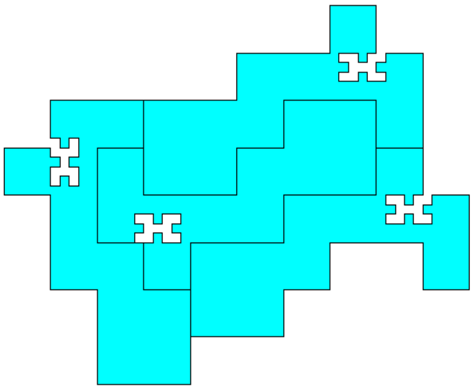

Figure 11. The tetrad with a prescribed single hole.

Minimal tetrads including as a single hole n-omino, if n = 1, 2, 3 and 4, have been shown in Figure 12. In this class of tetrads T-tetromino requires the largest tetrad made of 15-ominoes.

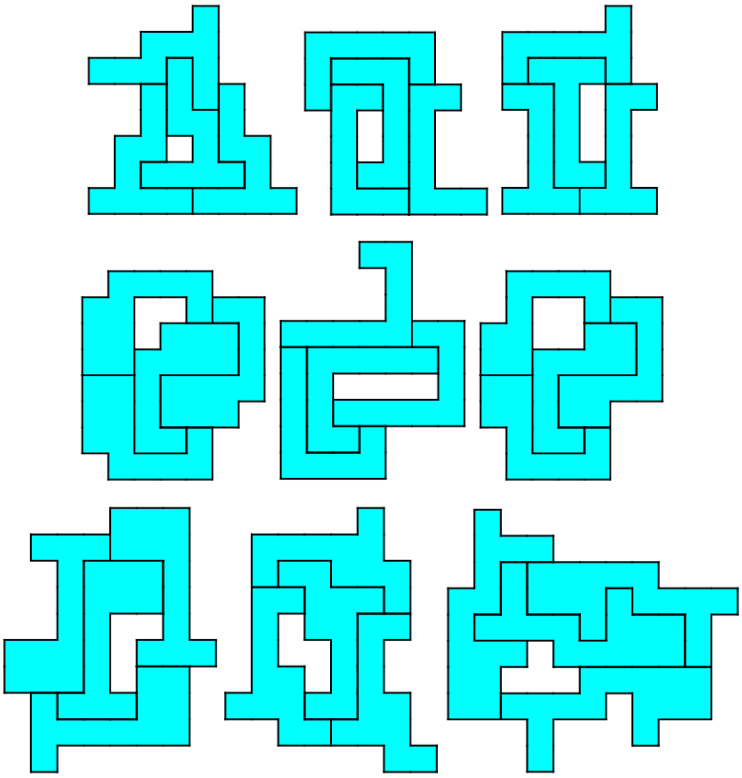

Figure 12. Minimal tetrads with a prescribed single hole.

With the help of a computer, the problem of including pentominoes in minimal tetrads has been completely solved. The greatest difficulties had to be overcome to find the minimal tetrad for pentomino X, see Figure 13, this pentomino requires the checking of all n-ominoes up to n = 17. The results are summarised in Table 2. The numbers of the second row of this table represent the minimum number of squares of the g-element.

Table 2. The number of the squares of g-elements.

I	**P**	**L**	**Z**	**W**	**F**	**N**	**T**	**U**	**V**	**Y**	**X**
10	**10**	**12**	**12**	**13**	**15**	**15**	**15**	**15**	**15**	**15**	**17**

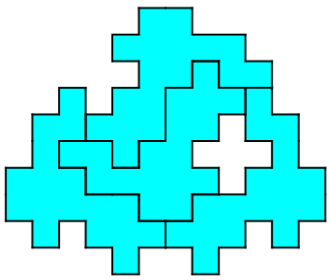

Figure 13. Minimal tetrad including the pentomino X.

Remark. As to including hexominoes there are 4 hexominoes (Figure 14) for which minimal tetrads have not been determined. To include these hexominoes at least 18-ominoes are needed.

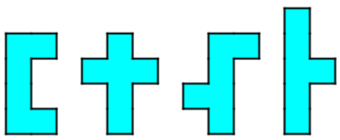

Figure 14. Four hexominoes.

5. Tritads

A tetrad has such an important property, one of its four components is always inside the tetrad, or one figure is surrounded by the other three. This property easy leads to the question: Can two identical shapes (polygons, polyominoes) include the same third shape (polygon, polyomino)? It is interesting that recreational mathematics deals with tasks that are essentially the construction of the tetrads, but a similar question with three figures, as far as we know, has been passed, maybe the authors of recreational mathematics may have believed that such figures do not exist at all. Rather unexpectedly, such figures exist and even in the polyomino class. For a general case see the paper (Mann, 2002). By analogy with the tetrads, we shall call such figures – **tritads**. The smallest tetrad uses 8-ominoes, but the smallest tritad – 12-ominoes, and unlike the tetrads it is unique, see Figure 15. There are two tritads made of 13-ominoes, both of them contain a hole of 9 unit squares, see Figure 16.

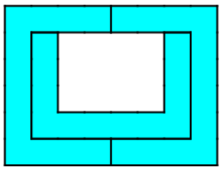

Figure 15. The minimal tritad.

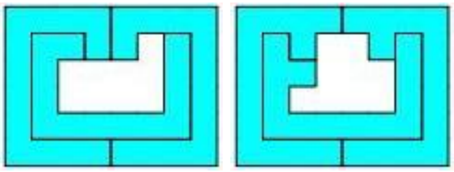

Figure 16. Tritads of 13-ominoes.

Computer calculations show that only these two tritads among all n-ominoes where $n \leq 19$, have the smallest known hole (9 unit squares). The results on tritads have been summarised in Table 3.

Table 3. The number of tritads.

n	**The number of useful *n*-ominoes**	**The number of tritads**
12	1	1
13	2	2
14	2	2
15	8	9
16	41	66
17	161	250
18	601	879
19	2006	3056

6. Some unsolved problems

P1. What is the smallest number of edges of g-element like? The answer is known for polyominoes, namely this number is 6, see Figure 1, and 8 for full tetrads, see Figure 4. In the book (Gardner, 1989) two full tetrads made of 6-gons are given, and the question what the smallest *n*-gon is like has been asked.

P2. The smallest known full tetrad (Figure 17) for a symmetric polyomino was found independently by Frank Rubin and Karl Scherer. It uses 34-ominoes (WEB, a). Is there a full tetrad of smaller *n*-ominoes with vertical symmetry? What symmetric *n*-ominoes are useful to form full tetrads? The tetrad found by Juris Čerņenoks uses 48-ominoes (Figure 18).

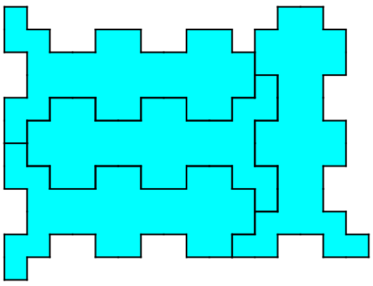

Figure 17.

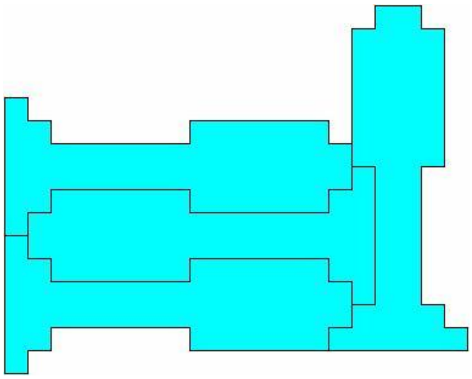

Figure 18.

P3. Does a full tritad exist? If it does not, what is the smallest possible hole like?

P4. Is there a symmetric full tetrad, the generating element of which is a symmetric polyomino? Does a full tetrad from *n*-omino with point (or birotary) symmetry exist? Does a full tetrad from *n*-omino with diagonal symmetry exist? Juris Čerņenoks found the smallest tetrads for polyominoes (having holes) with diagonal symmetry, which use 19-ominoes (WEB, a).

P5. Is there an integer *m* such that for all $n \geq m$ there exists a symmetric full tetrad made of *n*-ominoes?

P6. Is there a tetrad the g-element of which has two symmetry axes?
Two smallest known tetrads with g-element having one axis of symmetry are shown in Figure 19.

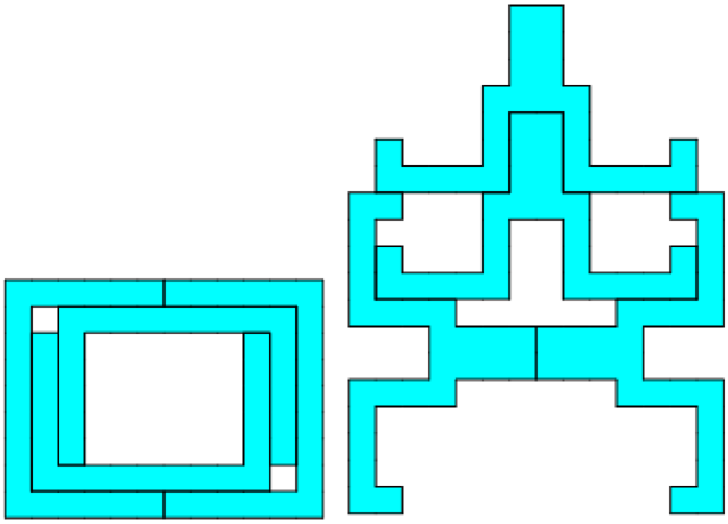

Figure 19. Symmetrical tetrads with symmetrical g-element.

References

Appel, K., Haken, W. (1977). *Every plane map is four colorable*, I, II, Illinois J. Math. 21, pp. 429-567.

Appel, K., Haken, W. (1977). *The Solution of the Four-Color-Map Problem*, Scientific American, Vol. 237, pp. 108-121.

Benjamin, A., Chartrand, G., Zhang, P. (2015). *The Fascinating World of Graph Theory*, Princeton University Press.

Cibulis, A., Čerņenoks, J. (2012), *Tetrads – it is not so easy*, MCG Newsletter 3, October, 2012, pp. 23-28. Available at http://www.brgkepler.at/~geretschlaeg/MCG_Newsletter_3.pdf

Cernenoks, J., Cibulis, A. (2013). *Application of IT in Mathematical Proofs and in Checking of Results of Pupils' Research*, Proceedings of the 6th International Conference on Applied Information and Communication Technologies, Jelgava, Latvia, pp. 172 - 177. Available at http://alephfiles.rtu.lv/TUA01/000040029_e.pdf

Dowek, G. (2015). *Computation, Proof, Machine, Mathematics Enters a New Age*, Cambridge University Press.

Gardner, M. (1989). *Penrose Tiles to Trapdoor Ciphers, Freeman*.

Mann, C. (2002). *A Tile with Surround Number* 2, The American Mathematical Monthly, Vol. 109, No. 4, pp. 383-388.

Redelmeier, D. H. (1981). *Counting Polyominoes: Yet Another Attack, Discrete Mathematics*, Vol. 36, pp. 191 – 203.

WEB (a) *Polyform Tetrads*, http://userpages.monmouth.com/~colonel/tetrads/tetrads.html

WEB (b) The On-line Encyclopedia of Integer Sequences, http://oeis.org/; https://oeis.org/A000105/list

Received June 19, 2018, accepted June 19, 2018