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On languages not recognizable by One-way Measure Many Quantum Finite automaton ⋆

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Abstract. Measure Many Quantum Finite Automaton is not the strongest known model of quantum automaton but, it is still an open question what is the class of languages recognized by it when only bounded error probability is allowed. In this paper a natural class of "word set" regular languages are studied. A proof that in general case the class can not be recognized by One-way model of Measure Many Quantum Finite automata is given.

Keywords: quantum computation, quantum automata, substring, MM-QFA

1 Introduction

Quantum finite automata (QFA) were introduced by several authors in different ways, and they can recognize different classes of languages. The least capable QFA are those (later called MO-QFA) introduced by C.Moore and J.Crutchfield [1]. More powerful notion of QFA (later called MM-QFA) was introduced by A.Kondacs and J.Watrous [2]. They introduced a 1-way and 2-way model of MM-QFA.

A.Ambainis and R. Freivalds[3] examined 1-way model and found that for some languages 1-way MM-QFA (1-QFA) can have exponential size advantages over classical counterparts. They used a language in a single-letter alphabet to prove these advantages. A.Ambainis and other authors[4] improved the base of the exponent in these advantages by using more complicated languages in richer alphabets. Later A.Ambainis together with N.Nahimovs [5] simpified the construction and improved the exponent even more. Nonetheless A.Kondacs and J.Watrous [2] showed that 1-QFA can only recognize regular languages, moreover, 1-QFA cannot recognize all the regular languages.

Future research had emphasis on 2-way automata because it was shown that they could recognize, all regular languages [6]. Also more general 1-way models of QFA where discovered[7][8], that could recognize all regular languages. Later more general models of quantum automata were introduced and studied[9].

There are still open problems for 1-QFA model. There are two models of 1-QFA: one that recognizes languages with bounded error probability and other

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that recognizes languages with unbounded error probability. It has been shown that unbounded model recognizes all and only stochastic languages[10]. It is not known what is the language class recognized by 1-QFA with bounded error[11]. The study of this class is interesting and challenging because it has been shown that it is not closed under union or any standard Boolean operation[12].

Due to these facts the study of different natural language classes recognizable by 1-QFA is reasonable and justifiable even if stronger models of quantum automata exist. In this paper we study a natural class of "word set" languages. We show that in general case it cannot be recognized by 1-QFA with bounded error.

2 Preliminaries

2.1 1-Way Measure Many Quantum Finite Automata(1-QFA)

We define 1-QFA as follows[13].

Definition 21 *A one-way quantum finite automaton $\mathcal{A}$ is specified by the finite (input) alphabet Σ, the finite set of states Q, the initial state q_0, the sets $Q_a \subseteq Q$ and $Q_r \subseteq Q$ of accepting and rejecting states, respectively, with $Q_a \cap Q_r = empty - set$, and the transition function*

$$\delta : Q \times \Gamma \times Q \to \mathbf{C}_{[0,1]},$$

where $\Gamma = \Sigma \cup \{\#, \$\}$ is the tape alphabet of $\mathcal{A}$, and $\#$ and $\$$ are endmarkers not in Σ. The evolution (computation) of $\mathcal{A}$ is performed in the inner-product space $l_2(Q)$, i.e., with the basis $\{|q\rangle | q \in Q\}$, using the linear operators V_σ, $\sigma \in \Gamma$, defined by

$$V_\sigma(|q\rangle) = \sum_{q' \in Q} \sigma(q, \sigma, q')|q'\rangle,$$

which are required to be unitary.

In computation of $\mathcal{A}$ the so-called computational observable $\mathcal{O}$ is used that corresponds to the orthogonal decomposition

$$l_2(Q) = E_a \oplus E_r \oplus E_n,$$

where $E_a = span\{|q\rangle | q \in Q_a\}$ and $E_r = span\{|q\rangle | q \in Q_r\}$ and E_n is the orthogonal complement of $E_a \oplus E_r$. Denote by P_p,$p \in \{a, r, n\}$ the projection operator into the subspace E_p.

A computation of A on an input $\#\sigma_1 \ldots \sigma_n\$$ proceeds as follows. The operator $V_\#$ is first applied to the starting configuration $|q_0\rangle$ and then the observable $\mathcal{O}$ is applied to the resulting configuration $V_\#|q_0\rangle$. This observable projects $V_\#|q_0\rangle$ into a vector $|\psi'\rangle$ of one of the subspaces E_a, E_r, E_n, with the probability equal to the square of the norm of $|\psi'\rangle$. If $|\psi'\rangle\ inE_a$ the input is accepted; if $|\psi'\rangle \in E_r$the input is rejected. If $|\psi'\rangle \in E_n$, then, after the normalization of $|\psi'\rangle$, the operator $V_{\sigma 1}$ is applied to $|\psi'\rangle$ and after that the observable $\mathcal{O}$ to the resulting vector. This

process goes on. Operators $V_{\sigma 1}, V_{\sigma 2}, \ldots, V_{\sigma n}$ are applied one after another, and after each such application the measurement by the observable $\mathcal{O}$ is performed. In all cases the computation continues only if a projection into E_n occurs.
When no termination occurs the computation can be seen as an application of the composed operator

$$V'_{\sigma n} V'_{\sigma n-1} \ldots V'_{\sigma 1} |q_0\rangle,$$

where $V'_{\sigma 1} = P_n V_{\sigma 1}$.
In order to define formally the overall probability with witch an input is accepted (rejected) by a 1QFA $\mathcal{A}$, we define the set $V_{\mathcal{A}} = l_2(Q) \times \mathbf{C} \times \mathbf{C}$ of so-called "total states" of $\mathcal{A}$, that will be used only with the following interpretation. $\mathcal{A}$ is at any time during the computation in the state (ψ, p_a, p_r) if so far in its computation $\mathcal{A}$ accepted the input with probability p_a, rejected with probability p_r and neither with probability $1 - p_a - p_r = \|\psi\|^2$, and $|\psi\rangle$ is its current, unnormalized state.
For each $\sigma \in \Gamma$ the evolution of $\mathcal{A}$, with respect to the total state, on an input σ is given by the operator T_σ defined by

$$T_\sigma(\psi, p_a, p_r) \rightarrow (P_n V_\sigma \psi, p_a + \|P_a V_\sigma \psi\|^2, p_r + \|P_r V_\sigma \psi\|^2).$$

For $x = \sigma_1 \sigma_2 \ldots \sigma_n \in \Gamma^*$ let $T_{\#x\$} = T_\$ T_{\sigma n} T_{\sigma n-1} \ldots T_{\sigma 1} T_\#$. If $T_{\#x\$}(|q\rangle, 0, 0) = (\psi, p_a, p_r)$, then we say that $\mathcal{A}$ accepts x with probability p_a and rejects with probability p_r. A 1QFA $\mathcal{A}$ is said to accept a language L with probability $\frac{1}{2} + \epsilon, \epsilon > 0$, if it accepts any $x \in L$ with probability at least $\frac{1}{2} + \epsilon$ and rejects any $x \notin L$ with probability at least $\frac{1}{2} + \epsilon$. If there is an $\epsilon > 0$ such that $\mathcal{A}$ accepts L with probability at least $\frac{1}{2} + \epsilon$,then L is said to be accepted by $\mathcal{A}$ with bounded-error probability. L is said to be accepted with unbounded error probability if $x \in L$ is accepted with probability at least $\frac{1}{2}$ and $x \notin L$ rejected with probability at least $\frac{1}{2}$.
On $V_{\mathcal{A}}$ we define a "norm" $\|\cdot\|_u$ as follows

$$\|(\psi, p_a, p_r)\|_u = \frac{1}{2}(\|\psi\| + |p_a| + |p_r|)$$

and let $D = \{v \in V_{\mathcal{A}} | \|v\|_u \leq 1\}$. D contains all global states of $\mathcal{A}$.

2.2 A useful lemma

Consider the following lemma

Lemma 21 *If $|u\rangle$ and $|v\rangle$ are vectors such that for a linear operator A, reals $0 < \epsilon < 1$ and $\mu > 0$,$\|A(u - v)\| < \epsilon$, and $\|v\|$, $\|u\|$,$\|Au\|$, $\|Av\|$ are in $[\mu, \mu + \epsilon]$, then there is a constant c, that does not depend on ϵ, such that $\|u - v\| < c\epsilon^{\frac{1}{4}}$.*

For proof look in [2]

3 Recognition of "word set" language class

Given a finite alphabet A and a finite set of words in this alphabet $\{w_1, w_2, \ldots, w_k\}$, consider the language class that consists of finite words $\{w_1, w_2, \ldots, w_k\}^*$.

Due to the properties of nondeterministic automata(NFA) it is easy to see that a NFA that recognizes language from this class can easily be constructed. Language that is recognizable by NFA is regular, so languages in "word set" language class are regular as well.

Theorem 31 *Given a language $S = \{w_1, w_2, \ldots, w_k\}^*$ it is not possible in general case to build a 1-QFA that recognizes this language.*

Proof. Lets look at the language $S_1 = \{x | x \in [\{0,1\}^6 1] \cup [\{0,1\}^8 1]\}^*$ and the length of the words are greater than 56 (so that each word length could be put together by parts of length 7 and 9) in other words, language consists of all finite words of length $l = 7 * a + 9 * b$ that end with 1 and other symbols are from $\{0, 1\}$.
It is easy to see that this language consists of subset of words from $\{0,1\}^*1$ language. Restrictions for determining the subset are: the word length is grater than 56 and there is no substring that consists of more than 8 following zeros. A.Kondacs and J.Watrous [2] state that:

Theorem 32 *The regular language $L_0 = \{0,1\}^*0$ cannot be recognized by a 1QFA with bounded-error probability.*

Proof. The proof is by contradiction. Let $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, Q_a, Q_r \rangle$ be a 1QFA recognizing the language L_0. To each $x = \sigma_1 \ldots \sigma_n \in \Gamma^*$ we assign the state

$$|\psi_x\rangle = V'_{\sigma n} \ldots V'_{\sigma 1} |q_0\rangle$$

and let $\mu = inf_{w \in \{0,1\}^*}\{\| \psi_{\#w} \|\}$. For each $w \in \{0,1\}^*$, $w0 \in L_0$ and $w1 \notin L_0$. If $\mu = 0$, then clearly $\mathcal{A}$ cannot recognize L_0 with bounded-error probability $\frac{1}{2} + \epsilon$. Let us therefore assume that $\mu > 0$. For any $\epsilon > 0$ there is a w such that $\|\psi_{\#w}\| < \mu + \epsilon$, and also $\|\psi_{\#wy}\| \in [\mu, \mu + \epsilon]$ for any $y \in \{0,1\}^*$. In particular, for any $m > 0$

$$\|V_1'^m |\psi_{\#w0}\rangle\| \in [\mu, \mu + \epsilon). \tag{1}$$

This implies that the sequence $\{V_1'^i |\psi_{\#w0}\rangle\}_{i=0}^{\infty}$ is bounded in the finite dimensional inner-product space and must have a limit point. Therefore there have to exist j and k such that

$$\|V_1'^j (|\psi_{\#w0}\rangle - V_1'^k |\psi_{\#w0}\rangle)\| < \epsilon.$$

By 21 last inequality together with 1 imply, that there is a constant c which does not depend on ϵ and such that

$$\||\psi_{\#w0}\rangle - V_1'^k |\psi_{\#w0}\rangle\| < c\epsilon^{\frac{1}{4}}.$$

This implies that

$$\|T_{\#w0\$}(|q_0\rangle, 0, 0) - T_{\#01^k\$}(|q_0\rangle, 0, 0)\|_u < c'\epsilon^{\frac{1}{4}}$$

for fixed c'. However, this has to be valid for an arbitrarily small ϵ. This is not possible if $\mathcal{A}$ accepts L_0 because $\mathcal{A}$ should accept the string $w0$ and reject $w01^k$. Hence $\mathcal{A}$ cannot accept L_0 with bounded error probability.
End of proof

From the proof we see that 1-QFA "has problems" recognizing the last 1 of the word, and that other characters are not of an importance. As a result - this theorem also apply to the language S_1. Language S_1 in general case is not recognizable by 1-QFA.QED
End of proof

4 Conclusion

In this paper a One way Measure Many Quantum Finite Automata model was studied. Although there are more general and stronger quantum automata models known, it is still an open question what is the class of languages that this model recognizes.

After studying a "word set" language class a theorem has been proven that there exists a language from "word set" language class that is not recognizable by 1-QFA.

The method of construction of the language is rather interesting, and there is a place for future research to look for specific word sets that can create a 1-QFA recognizable language and perhaps make the size of the minimal 1-QFA recognizing such language exponentially more efficient than deterministic finite automata.

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