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Domino Exclusion Problem

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Abstract. The classic domino exclusion problem consists of finding minimum number $d(n)$ of dominoes on an $n \times n$ chessboard to prevent placement of another domino. This sequence of minimum numbers is discussed under A280984 at the On-Line Encyclopedia of Integer Sequences. With new theoretical insights and a specially designed computer program we were able to expand the sequence from n =18 to n = 33. New upper bounds of $d(n)$ thought to be sharp have been obtained. The article also discusses the rectangle-free minimal domino packings. Small 3-dimensional grid squares up to n = 6 have been analysed.

Keywords: computer-assisted proof, domino, estimate, grid rectangle, matchstick, upper bound.

1. Introduction

During the Covid-19 pandemic, the first author was writing a book on recreational mathematics. In one chapter, both old and new tasks about matchsticks were collected. In relation to the number 19, the following task was devised: ”To create a vaccine that prevents Covid-19 virus from multiplying, it is necessary to colour a minimum number of unit edges of the 19×19 cell square so that each uncoloured edge has at least one point of contact with the coloured edge.” As it turned out later this rather difficult task had far-reaching consequences. Solving this problem for small matchstick squares $n \times n$ the following sequence of minimum numbers was obtained, see Figure 1:

2, 3, 6, 9, 12, 17, ...

Then, looking at The On-Line Encyclopedia of Integer Sequences (Shepard, 2017), it was understood that this matchstick problem is equivalent to the domino exclusion problem studied earlier in (Gyárfás et al., 1988). Domino exclusion problem consists of finding minimum number of dominoes on an $n \times n$ board to prevent placement of another domino, see Figure 2 as a transformation of Figure 1 in domino *terminology*.

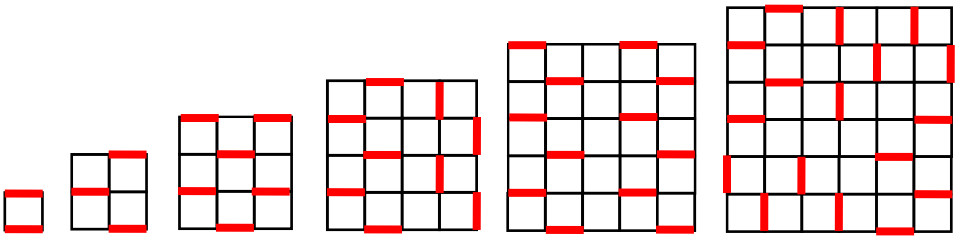

Figure 1. Minimum number of coloured sticks

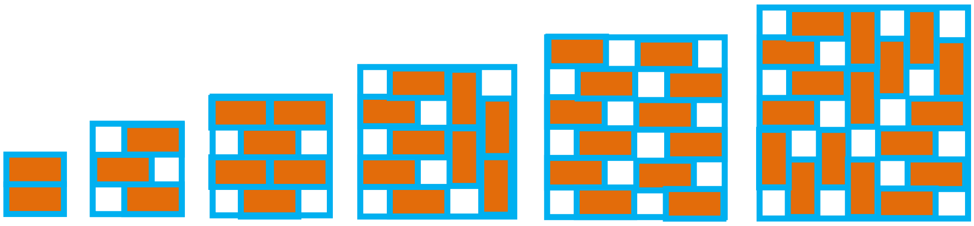

Figure 2. Minimum number of dominoes

A third way to visualize the problem and not to draw unpainted square edges at all is to use graphs, see Figures 3 – 4.

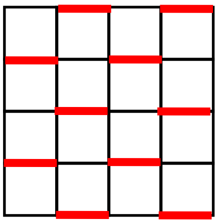

Figure 3. Non minimal arrangement with 10 sticks

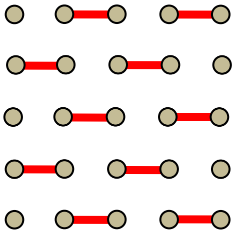

Figure 4. Interpretation by graph

As far as we know, the first book in which one can find the tasks of excluding shapes (namely, pentominoes by monominoes on a chessboard 8×8) is the Golomb's classic book (Golomb, 1994). Exclusion problems in other areas (graph theory, statistical physics, percolation theory) may have to do with the following concepts: matching, minimum dominating sets, domination number, square grid graphs, an edge cover, dimmers, and others (Alanko et al., 2011), (Korte and Vygen, 2018).

The second author developed an efficient algorithm that allowed new progress in both the domino exclusion problem and its generalizations to n-dimensional grids.

The article uses generally accepted terms in mathematics:

$$\lceil x \rceil = \min\{n \in Z,\, n \geq x\},\ \lfloor x \rfloor = \max\{n \in Z,\, n \leq x\}$$

are the so-called ceiling and floor functions of x, respectively.

2. Grid squares and rectangles

Let $[m \times n]$ be a grid rectangle consisting of *m* rows and *n* columns of points (dots, vertices, grid meshes), and let $D = D(m, n)$ be the minimum number of dominoes (edges isolating grid points) for which a packing exists. A domino packing is an arrangement of dominoes on a given board (here on a grid rectangle) to prevent placement of another domino. The following numbers are of particular importance in future estimates:

$$d(n) := D(n, n),\ \ D_0(m, n) := \left\lceil \frac{mn}{3} \right\rceil, m, n \geq 2,\ \ d_0(n) := D_0(n, n).$$

A point is isolated if all its neighbours are connected by edges. Note that minimizing the number of edges or maximizing the number of isolated points or *holes* $H = H(m, n)$ are equivalent problems. Clearly that

$$H + 2D = mn. \tag{1}$$

2.1. Estimates

Two important theorems are proved in this section. As a consequence of the first theorem, the following nice estimate is obtained:

$$D(m, n) \geq \left\lceil \frac{mn}{3} \right\rceil, m, n \geq 2. \tag{2}$$

Both old and new (updated) information one can found in (Kagey, 2019): "Fifteen terms are known, and a few folks have conjectured that

$$A2808984(n) = \left\lceil \frac{n^2}{3} \right\rceil \text{ for } n > 1. \tag{3}$$

Walter Trump has just added the terms 19 - 33 of the sequence (with $d(19) = 122 = \lceil 19 \cdot 19 / 3 \rceil + 1$ with some examples of optimal solutions and announced an effective algorithm for finding the optimal solutions. /Jun 14 at 20:30/"

The sequence of the left hand side of (3) here means the numbers $d(n)$, and since $d(19) \neq d_0(19)$, the conjecture (3) is generally incorrect. Here, ironically, numerologists should have known that Covid-19 number breaks for the first time this beautiful formula. Life is not so simple and we have to look for a new formula.

Such estimates of $d(n)$ are given in (Gyárfás et al., 1988):

$$d(n) \geq \left\lceil \frac{n^2}{3} \right\rceil, \quad d(n) \leq \frac{n^2}{3} + \frac{n}{12} + 1.$$

"If n is large and $n = 3k \pm 1$ then $d(n) \geq \frac{n^2}{3} + \frac{n}{111}$." (It is not specified how *large*).

Remark 1. $H \leq D + 1$ is valid for *strings* (*rectangles* with one row or one column). Equality can only exist if the number of grid points is $n = 3k + 1$, see Figure 5 with H = 4 and D = 3.

Figure 5. Domino arrangement with $H = D + 1$

For rectangles, a stronger estimate $H \leq D$ can be proved (Theorem 1). The proof uses the same basic ideas as in the article (Gyárfás et al., 1988) but the presentation is shorter and quite different.

Theorem 1. $H(m, n) \leq D(m, n),\ m, n \geq 2.$

Proof. Let us use the following notations**:**

B – set of boundary points of $[m \times n]$ rectangle (points in the 1st or last row, or column)

D, H – the number of dominoes, holes respectively,

D_B, H_B – the number of dominoes, holes that are incident (have contact, touch) to B,

$D_j, H_j,\ j = 1, 2, 3, 4$ – the number of dominoes, respectively holes incident to the j-th side of the rectangle, see Figure 6 where

$D_1 = D_3 = 2, D_2 = D_4 = 2, H_1 = H_3 = 3, H_2 = H_4 = 3,$

$T_j := D - D_j + H_{2+j} - H,\ \ H_5 := H_1,\ \ H_6 := H_2,\ \ j = 1, 2, 3, 4,$

$T := T_1 + T_2 + T_3 + T_4.$

Such equalities:

$$D = D_j + H - H_{2+j} + T_j,\ j = 1, 2, 3, 4, \tag{4}$$

follow immediately from the notations, but their essence, *salt*, is in their geometric interpretation. Let us interpret (4), e.g. for j = 1. The difference $H - H_3$ means the number of holes in the rectangle, except for the last row. Each such hole has a domino below it. It is important that each hole has its own corresponding domino. T_1 is the number of dominoes not yet counted. T_1 describes (*redundant*) dominoes exactly above which there are no holes and which do not *belong* to D_1. Figure 7 shows several domino arrangements in the form of tetrominoes I, L, and O which yield $T > 0$. (Here is a motivation for choosing the symbol *T* – tetromino.)

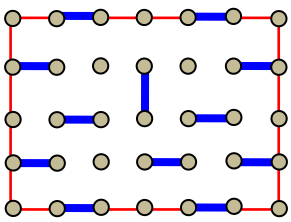

Figure 6. Example of $[5 \times 7]$ packing

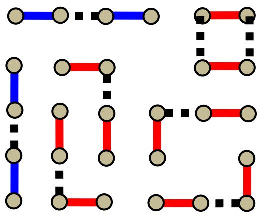

Figure 7. Arangements with $T > 0$

Without loss of generality we can assume that $H_1 \geq H_3$. Since $H_1 \leq D_1 + 1$ then (4) immediately yields:

$$D = D_1 + H - H_3 + T_1 \geq H + D_1 - H_1 + T_1 \geq H + T_1 - 1.$$

From here, $D \geq H$, if $H_1 \leq D_1$. Due to (1): if $H = D + 1$ then

$$3D + 1 = mn \Rightarrow mn = 1(\mathrm{mod}\,3).$$

Thus, it remains to examine only those rectangles for which

$$T_j = 0, H_j = D_j + 1, \ j = 1, 2, 3, 4, \ H = D + 1, \ mn = 1(\mathrm{mod}\,3). \qquad (5)$$

Assume *R* is the smallest rectangle that satisfies (5). The number of *B* points is an even number, so they can be grouped in pairs. Since at least one point in each pair is the domino endpoint, then $H_B \leq D_B$. If *B* would contain only dominoes with both endpoints inside *B*, then after discarding *B* we would get a new rectangle smaller than *R* with $H \geq D + 1$ (and since $H > D + 1$ is not possible then $H = D + 1$), which contradicts the assumption. So there is a domino that has only one endpoint in *B*. Without loss of generality, we can assume that this is the horizontal domino with one point in the first column: $(i, 1), i \neq 1, n,$ see Figure 8. Then the points $(i \pm 1, 1)$ are holes (otherwise the condition $H_4 = D_4 + 1$ would not be fulfilled), but their adjacent points $(i \pm 1, 2)$ are endpoints of horizontal dominoes. A vertically placed domino in points $(i \pm 1, 2)$ formed L-tetromino arrangement (Figure 7) giving $T > 0$. Since the number of columns 3 is not valid and the point $(i, 3)$ is a hole (otherwise $T > 0$) the points $(i, 4)$ and $(i, 5)$ are joined by domino. Points $(i \pm 1, 4)$ are holes (otherwise $T > 0$). Thus only the arrangements shown in Figure 9 is permissible. Since the process is not complete in the column $m \neq 3k$, the theorem is proved.

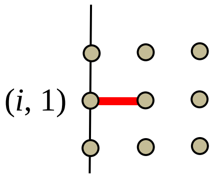

Figure 8.

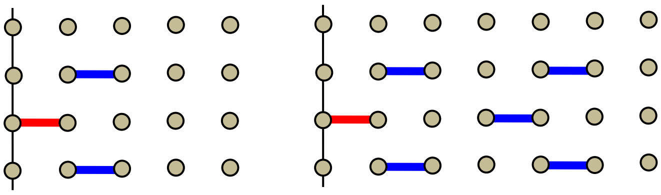

Figure 9. Periodic arrangements

Remark 2. With this proof technique, an estimate $H \le D$ can also be obtained for 3-dimensional grid rectangles.

Corollary. $D(m, n) \ge D_0(m, n)$, see (2).

Proof: $(H + 2D = mn, H \le D) \Rightarrow 3D \ge mn \Rightarrow D \ge \dfrac{mn}{3} \Rightarrow D \ge \left\lceil \dfrac{mn}{3} \right\rceil$ because D is an integer.

Theorem 2. The following estimates are valid:

$$d(n) \le d_0(n), \;\; n = 3k, \tag{6}$$

$$d(n) \le d_0(n) + \left\lfloor \frac{n-3}{15} \right\rfloor, \;\; n = 3k+1, \tag{7}$$

$$d(n) \le d_0(n) + \left\lfloor \frac{n}{21} \right\rfloor, \;\; n = 3k+2. \tag{8}$$

Proof

1. The case $n = 3k$, estimate (6), is trivial in the sense that minimal arrangements (solutions) also for $[m \times n]$ rectangles one can obtain using an *elementary* pattern shown in Figure 10. Copies of $[m \times 3]$ can be added to each other as many times as needed. Moreover, it is important that the additive property holds:

$$D_0(m_1 + m_2, 3k) = D_0(m_1, 3k) + D_0(m_2, 3k).$$

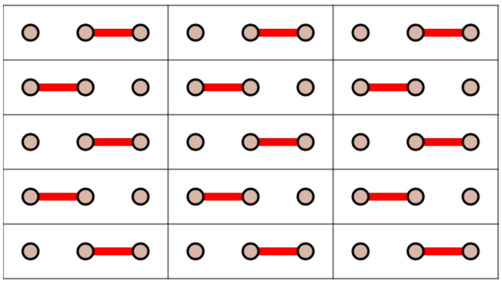

Figure 10. Periodic arrangement

2. Case $n = 3k + 1$ is the most difficult to prove.
Let us compose the $[n \times n]$ square from *elementary* blocks: $[1 \times n]$ and $[4 \times n]$ rectangles, see Figure 11 with n = 4, 7, 10.

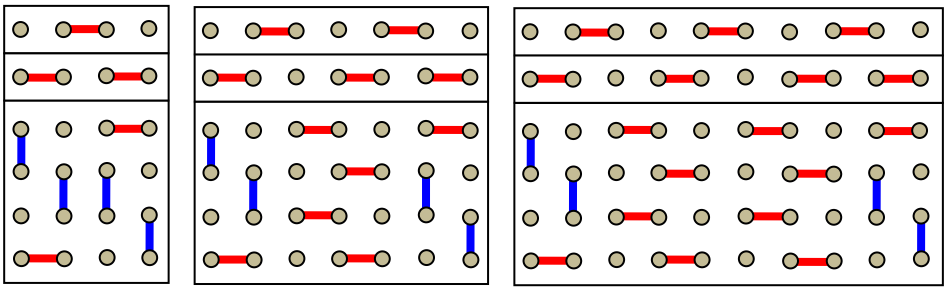

Figure 11. Elementary blocks for forming squares with $n = 3k + 1$

It is clear that each such block can be stretched to a length $n = 3k + 1$. Suppose we have x rectangles $[1 \times n]$ and y rectangles $[4 \times n]$. Then $x + 4y = n$. There are two types of $[1 \times n]$ rectangles with the following number of coloured edges: $\frac{n-1}{3}$, and $\frac{n+2}{3}$, respectively. If these two types of rectangles are adjacent and form $[2 \times n]$ then

$$D(2, n) = \frac{2n+1}{3} = 2k + 1. \tag{9}$$

The number of coloured edges for $[4 \times n]$ rectangles is $D(4, n) = \left\lceil \frac{4n}{3} \right\rceil$. Note that two $[4 \times n]$ rectangles are not adjacent, because in that case we would not get the minimal arrangement:

$$2\left\lceil \frac{4(3k+1)}{3} \right\rceil = 8k + 4 > \left\lceil \frac{8(3k+1)}{3} \right\rceil = 8k + 3.$$

Let's choose the following number of $[4\times n]$ rectangles: $y=\left\lfloor\frac{n+1}{5}\right\rfloor$. The motivation for such a choice is that length 4 is followed by length 1 and we want to use the maximum number of such rectangles. For example, if $n=19$, then $y=4$, and partition of 19 is as follows: $19=4+1+4+1+4+1+4=4\cdot 4+3\cdot 1$. For simplification purposes, let us use the following elementary calculations:

$$\left\lceil\frac{4n}{3}\right\rceil=\left\lceil\frac{4(3k+1)}{3}\right\rceil=4k+2;$$

$$\left\lceil\frac{n^2}{3}\right\rceil+\left\lfloor\frac{n-3}{15}\right\rfloor=3k^2+2k+1+\left\lfloor\frac{3k-2}{15}\right\rfloor;\, nk=3k^2+k.$$

2a. If all $[1\times n]$ rectangles contain k coloured edges, then (since $x+4y=n$)

$$d(n)\le y(4k+2)+xk=kn+2y\le\left\lceil\frac{n^2}{3}\right\rceil+\left\lfloor\frac{n-3}{15}\right\rfloor \Leftrightarrow$$

$$2\left\lfloor\frac{3k+2}{5}\right\rfloor\le k+1+\left\lfloor\frac{3k-2}{15}\right\rfloor. \quad (10)$$

The correctness of this inequality can be easily proved by taking $k=5j+r$ and checking the five values of remainder r.

2b. If all $[1\times n]$ rectangles except one contain k coloured edges then, see (9),

$$d(n)\le y(4k+2)+(x-2)k+2k+1=y(4k+2)+xk+1,$$

and instead of (10) we now have the inequality

$$2\left\lfloor\frac{3k+2}{5}\right\rfloor\le k+\left\lfloor\frac{3k-2}{15}\right\rfloor. \quad (11)$$

This inequality for arbitrary k is not correct at all, for example, $k = 3$. But here there is a subtle nuance, namely, the inequality does not have to be checked for all k, but only for those k for which $n = 3k + 1$ does not fit in the case **2a**. Note that only such k values need to be checked: $k=5j+r$, where $r = 2$ or 4. It is easy to prove that the inequality (11) for both values of r transforms into equality.

3. Case $n=3k+2$. Let us compose the $[n\times n]$ grid square from x rectangles $[1\times n]$ and y rectangles $[3\times n]$ with

$$D(1,n)=\frac{n+1}{3},\, D(3,n)=n.$$

See Figure 12 with such *elementary* rectangles for n = 20. Let's choose the following number of $[1\times n]$ rectangles: $x = 2 + 3\left\lfloor \frac{k}{7} \right\rfloor$.

Since $x + 3y = n$, then

$$d(n) \le \frac{x(n+1)}{3} + yn = \frac{xn+x}{3} + \frac{(n-x)n}{3} = \frac{x}{3} + \frac{n^2}{3} = \frac{2}{3} + \left\lfloor \frac{k}{7} \right\rfloor + \frac{n^2}{3} =$$

$$= \frac{2+n^2}{3} + \left\lfloor \frac{k}{7} \right\rfloor \le \left\lceil \frac{n^2}{3} \right\rceil + \left\lfloor \frac{3k+2}{21} \right\rfloor \Leftrightarrow \left\lfloor \frac{k}{7} \right\rfloor \le \left\lfloor \frac{k}{7} + \frac{2}{21} \right\rfloor.$$

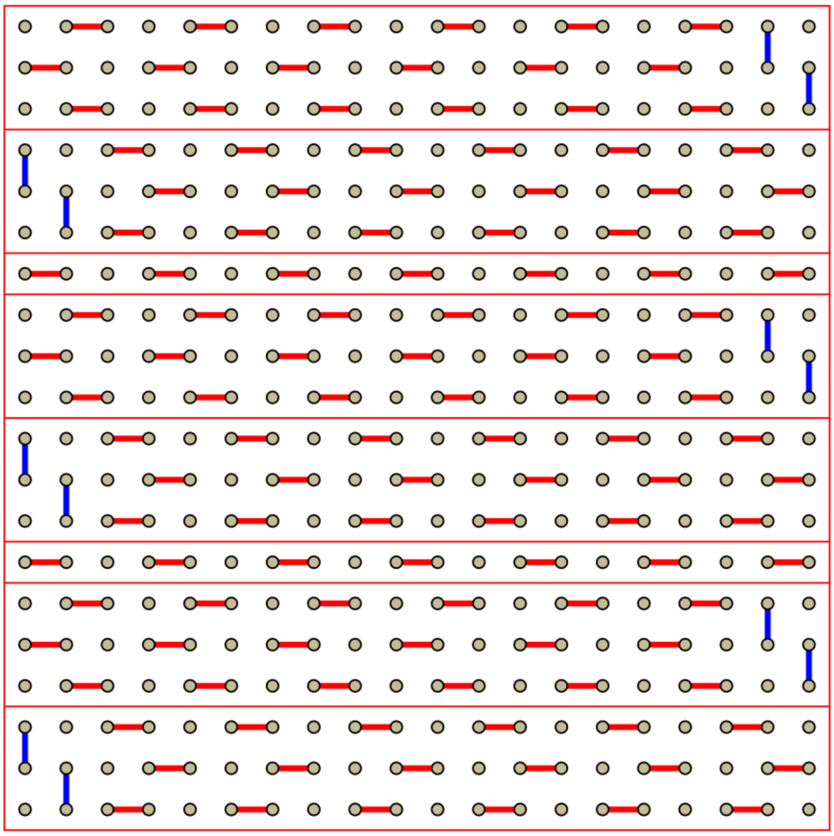

Figure 12. Minimal arrangement, $d(20) = 134$

2.2. Rectangles with $D(m, n) = D_0(m, n)$

We have previously found that the smallest square for which equality (12) does not hold is $[19 \times 19]$. In this section we will look for the smallest rectangle for which the analogue of equality (12), i.e. (13) is no longer valid.

$$D(n, n) = D_0(n, n) = d_0(n) = \left\lceil \frac{n^2}{3} \right\rceil \quad (12)$$

$$D(m, n) = D_0(n, m) = \left\lceil \frac{nm}{3} \right\rceil. \quad (13)$$

Theorem 3. $D(m, n) = \left\lceil \frac{mn}{3} \right\rceil$ for $2 \leq m \leq 13, n \geq 2.$

Proof. According to (2) it suffices to show solutions (arrangements) with the specified number $D_0(n, m)$ of dominoes. Such solutions are easy to find for small *m*. For *m* = 2, see Figure 13, and for *m* = 3*k*, see the periodic arrangement shown in Figure 10.

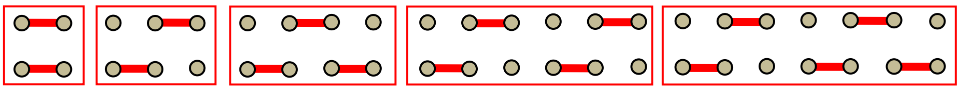

Figure 13. Minimal arrangements for *m* = 2

The fact that the presented solutions contain the required number of dominoes can be easily verified using the property:

$$D_0(m, j + 3k) = D_0(m, j) + D_0(j, 3k).$$

Periodically added rectangles is also of use for *m* = 4, and *m* = 5, see Figure 14. By repeating $[4 \times 3]$ and $[5 \times 3]$ rectangles the required number of times, we can get $[4 \times j + 3k]$, $[5 \times j + 3k]$, $j = 1, 2, 3,$ i.e. all necessary rectangles with $m = 4$ and *m* = 5.

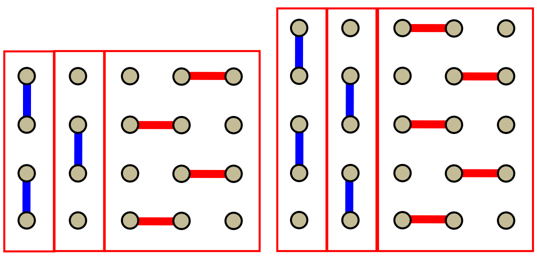

Figure 14. Minimal arrangements for *m* = 4 and *m* = 5

Now let's use a more advanced idea: in the role of *elementary* rectangles, let's take appropriately selected blocks that are periodically added. For m = 13 the minimal arrangements are shown in Figure 15. The key to the proof is now a periodically movable string (a block of red edges), which we move by three units to obtain all the required rectangles. From here, removing the first two rows, we easily get the minimal arrangements for rectangles with m = 11 rows. Similarly, removing the appropriate number of rows we will obtain minimal arrangements for the other required values for m (m = 10, 8, and 7). To avoid ambiguity, let us clarify that in the case $n = 3k + 5$, after removing the first two rows, the vertical domino is shifted down one unit, and after removing the first 5 rows, the vertical domino is replaced by a horizontal one. Theorem is proved.

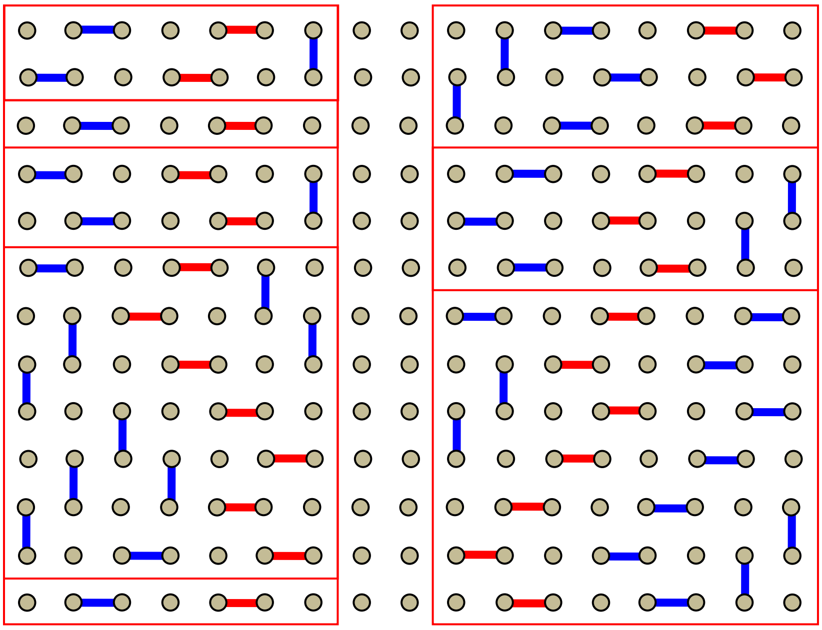

Figure 15. Minimal arrangements for m = 13, $n = 3k + 4$, $n = 3k + 5$

By computer assisted-proof it is stated that $[14 \times 16]$ is the smallest rectangle for which the formula (13) is no longer valid. The smallest rectangle of type $[m \times (m+1)]$, for which the formula is not valid, is $[16 \times 17]$. As for the rectangles $[14 \times n]$, $n = 2(\mathrm{mod}\,3)$, this formula is again correct, see Figure 16.

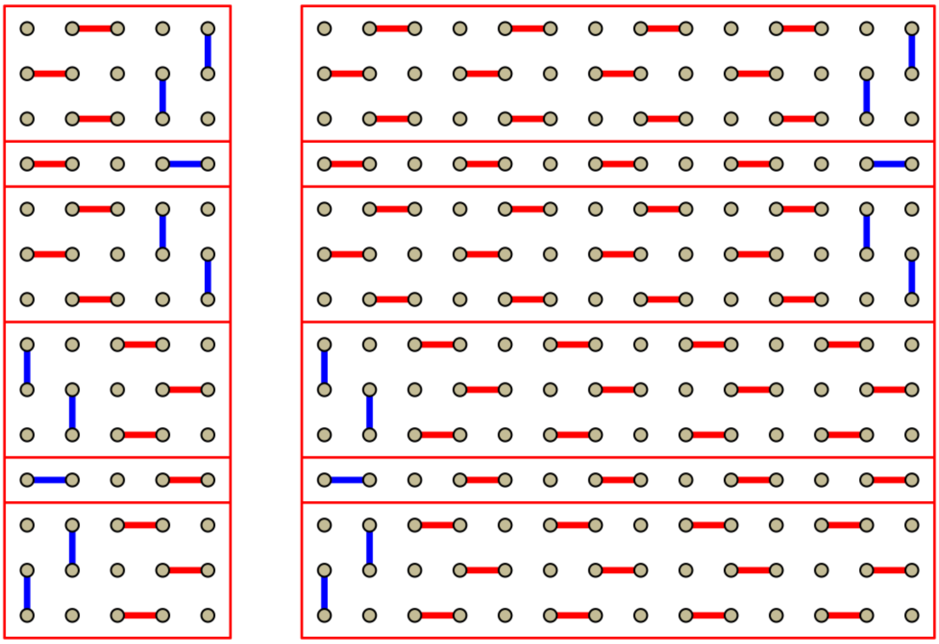

Figure 16. Minimal arrangements for $m = 14$, $n = 2 + 3k$

Remark 3. Analogous estimates for rectangles can also be obtained by a similar proof technique (as in Theorem 1). A more complicated estimate is for rectangles whose edge lengths divided by three give different remainders:

$$D(m, n) \leq \left\lceil \frac{mn}{3} \right\rceil + \min\left\{ \left\lfloor \frac{m+3}{15} \right\rfloor, \left\lfloor \frac{n+6}{21} \right\rfloor \right\}, \; m = 2(\mathrm{mod}\,3), \; n = 1(\mathrm{mod}\,3).$$

Remark 4. As a further study, we propose the following hypothesis:

$$d(n) = \begin{cases} \left\lceil \dfrac{n^2}{3} \right\rceil + \left\lfloor \dfrac{n-3}{15} \right\rfloor, & n = 3k+1, \\ \left\lceil \dfrac{n^2}{3} \right\rceil + \left\lfloor \dfrac{n}{21} \right\rfloor, & n = 3k+2. \end{cases}$$

2.3. Rectangle-free packings

In the previous section, the partition of squares in rectangles was crucial to prove the theorems. Let us now consider the question of the existence of minimal packings which cannot be divided into smaller rectangles. Such packings will be called rectangle-free packings (arrangements, solutions). The smallest square for which a rectangle-free packing exists is the $[5\times 5]$ square, the next such squares are $[8\times 8]$, $[10\times 10]$, $[11\times 11]$ and $[14\times 14]$, see Figures 17 - 18. For $[3k\times 3k]$ with $k\geq 3$ there exist exactly 16 different packings if reflected and rotated solutions were count. But up to symmetry there are only 3 packings, see Figure 19. No packing is rectangle-free.

With the help of a computer program, it has been found that rectangle-free packings with $d(n)=d_0(n)$ is a rarity in general, the largest square for which such a solution still exists is $[14\times 14]$. This is an unexpected result, at least for the first time, because larger squares no longer have this type of solution. For rectangles (unlike squares) the number of rectangle-free packings with $d(m,n)=d_0(m,n)$ is not finite, see, e.g. Figure 20 obtained from the $[5\times 5]$ square repeating the red fragment. The fact that there is no a rectangle-free packing for $[n\times n]$ square does not mean that there is no a rectangle-free packing for $[m\times n]$ rectangle. See Figure 21 as an example.

Developing the idea of periodicity in two directions, we manage to find rectangle-free packings of squares with $d(n)$ satisfying (7) or (8) for arbitrarily large rectangles, see Figure 22. So far rectangle-free minimal packings with $d(n)=d_0(n)+1$ are known for n = 19, 22, 23, 26. See Figure 23 for n = 26.

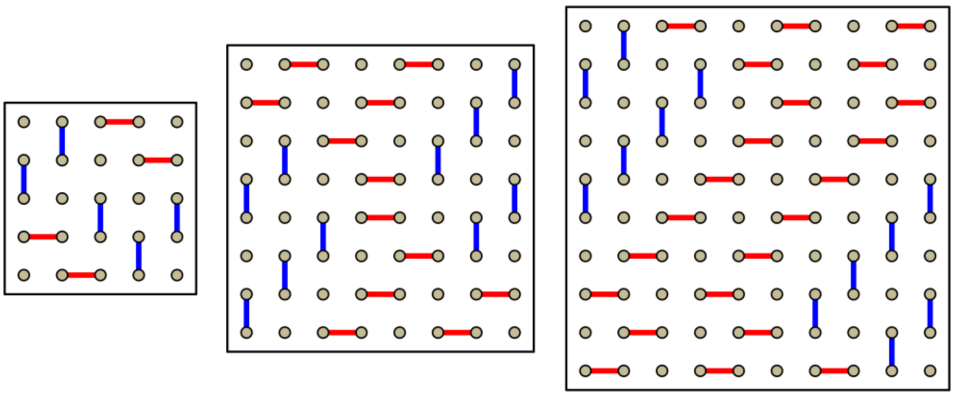

Figure 17. Rectangle-free packings: $d(5)$ = 9, $d(8)$ = 22, $d(10)$ = 34

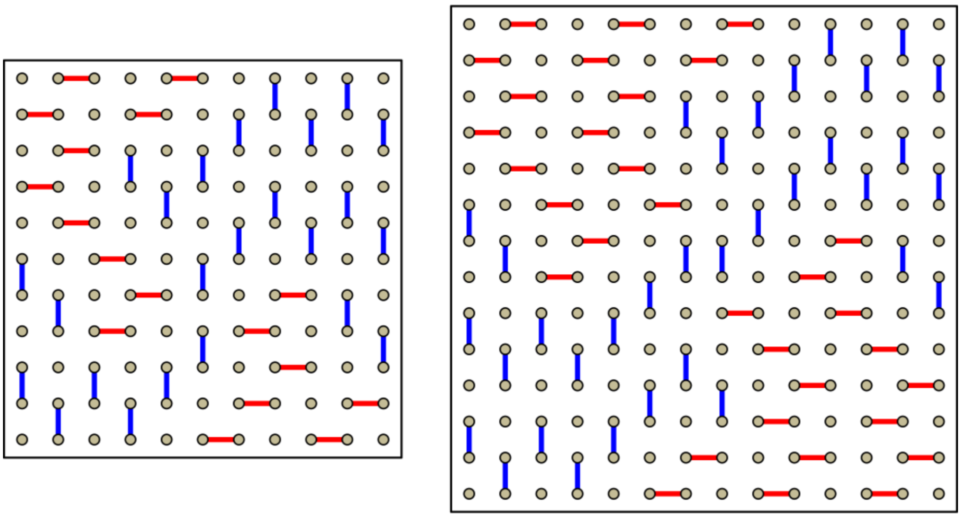

Figure 18. Rectangle-free packings, $d(11) = 41$, $d(14) = 66$

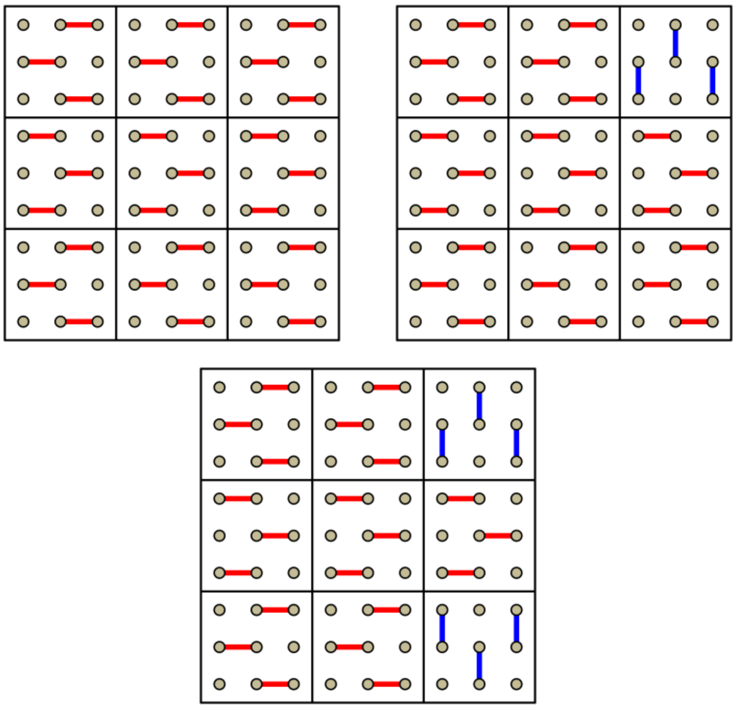

Figure 19. Trivial minimal packings, $n = 3k$

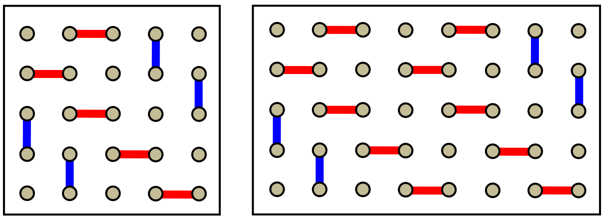

Figure 20. Rectangle-free minimal packings, $m = 5$, $n = 2 + 3k$

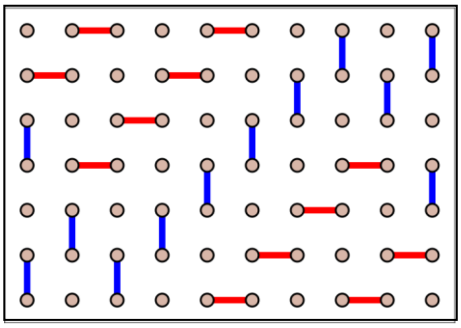

Figure 21. Rectangle-free packing for $[7 \times 10]$

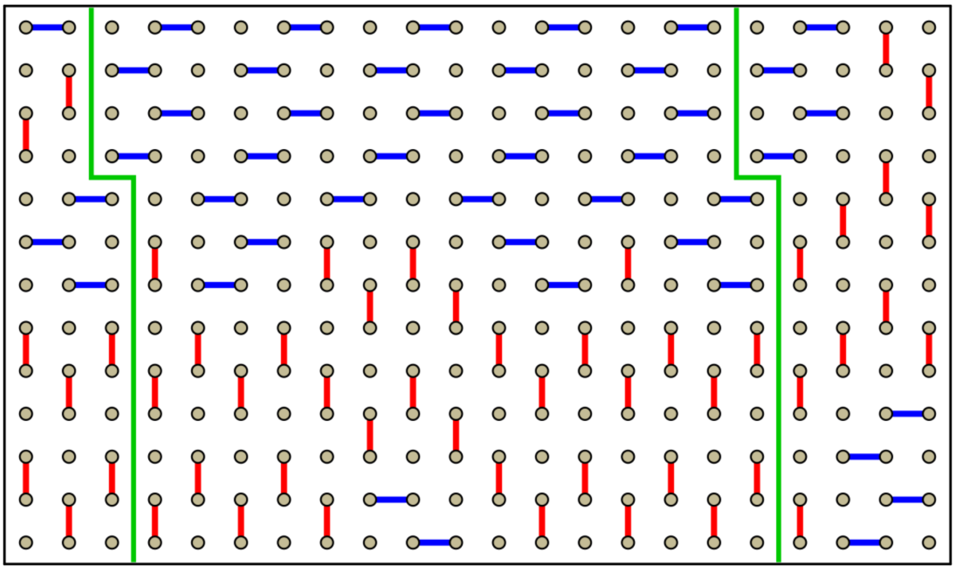

Figure 22. Rectangle-free packing for $m = 13 + 3j$, $n = 7 + 15k$

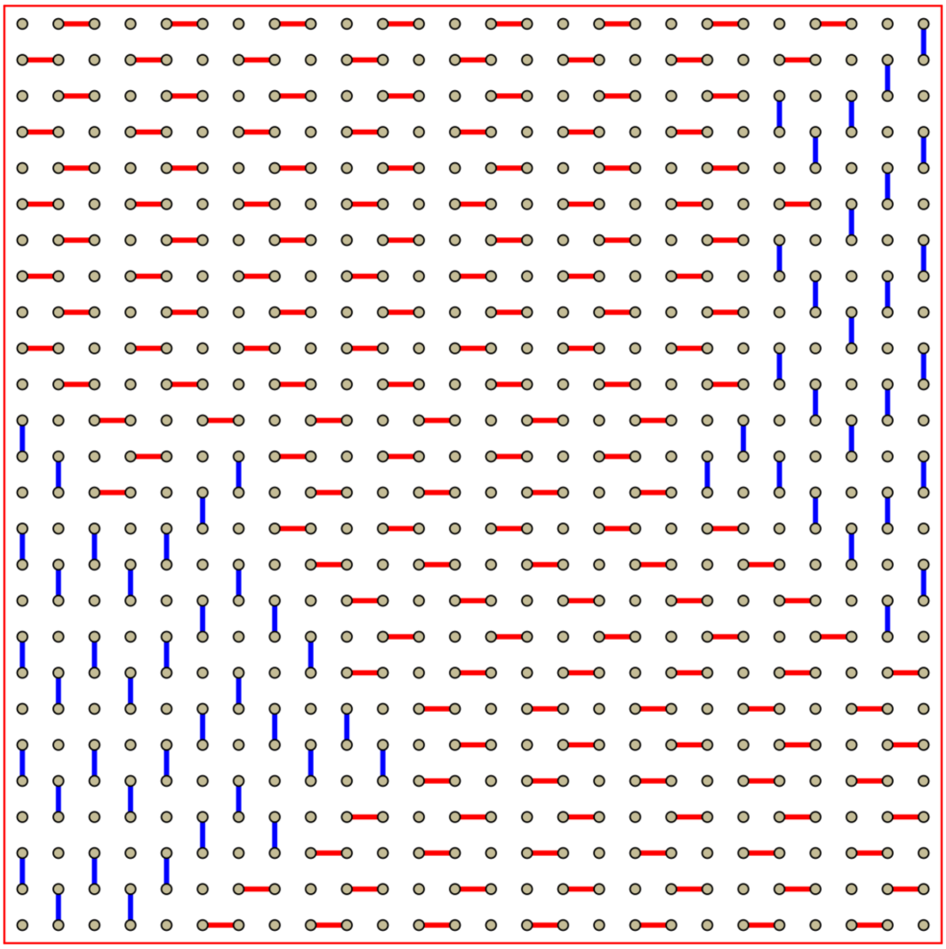

Figure 23. Rectangle-free packing, $d(26) = 227$

3. Basic information for algorithm elaboration

In this section we use the following notations:

D_L, D_R – number of dominoes which cover at least one cell in the left (the first), respectively right (the last) column,

H_L, H_R – number of holes in the left respectively right column.

T_L, T_R – number of dominoes which touch only dominoes with their left, respectively right edge. (The contact line is one unit for horizontal dominoes and 2 units for vertical dominoes.)

$M_L := D_L + 1 - H_L, M_R := D_R + 1 - H_R$ – number of *missing* holes

$(H_L \le D_L + 1, H_R \le D_R + 1 \Rightarrow M_L \ge 0, M_R \ge 0)$

$B := T_L + T_R + M_L + M_R$ – number of *bad* domino constellations,

$$2(D - H) = T_L + T_R + D_L - H_L + D_R - H_R \tag{14}$$

$$2(D + 1 - H) = T_L + T_R + M_L + M_R = B. \tag{15}$$

Equality (14) with precision to the notations is equivalent to equality (7) from (Gyárfás et al., 1988). Equality (15) immediately follows from (14). Since $mn = 2D + H$, then $H = mn - 2D$ and

$$2(D + 1 - H) = 2(3D + 1 - mn) = 6D + 2(1 - mn) \Rightarrow$$

$$6D = 2(mn - 1) + B. \tag{16}$$

Equality (16) is very important. It shows that by minimizing *B* we minimize *D*.

With a backtracking algorithm we enumerate the domino packings of a $[m \times n]$ rectangle with a given number *D* of dominoes. We do this by placing dominoes row by row from left to right. In general there are 3 possibilities to continue in a grid cell: empty, horizontal or vertical domino. Therefore the number of paths is greater than $3^{mn/2}$. Even rather small rectangles cannot be handled as the number of paths is too high.

The new approach considers the known number *B* of *bad* domino constellations. As soon as (*B* + 1) such constellations are reached the current path can be abandoned. For small *B* this algorithm works very fast. Dependent on the used processor and programming language the enumeration (determination of the number of all packings for a number *D* of dominoes) for squares up to $[20 \times 20]$ can be done in less than a minute. The status of each cell is described in an oversized array sq(x,y) with 0 ≤ x ≤ n + 1 and 0 ≤ y ≤ m + 1, where x is the column and y the row, with the following numerical characteristics (Figure 24):

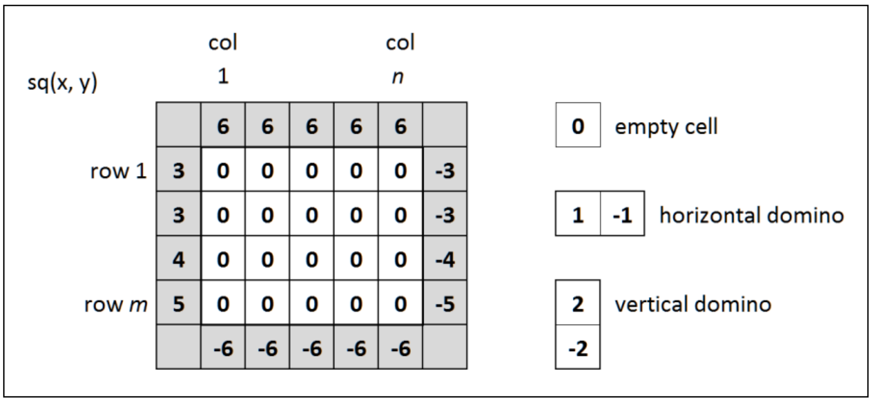

Figure 24. Numerical characteristics

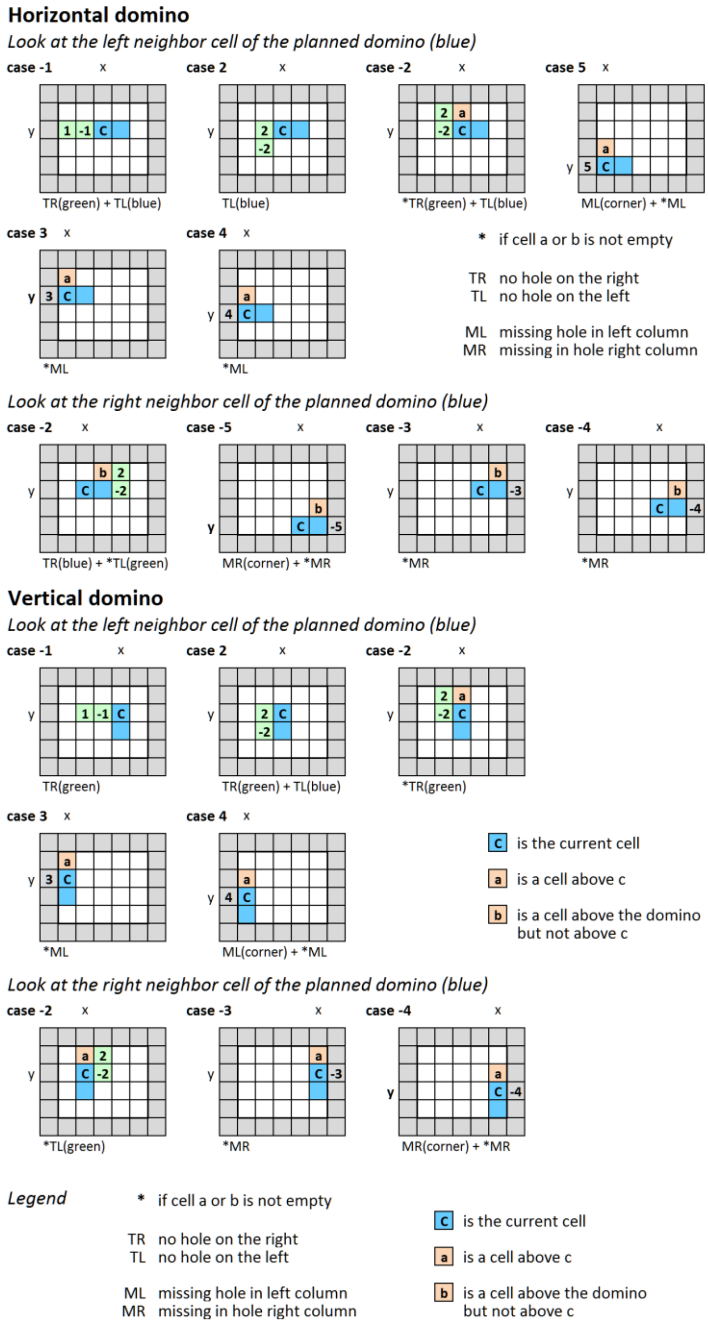

Figure 25. Different cases of domino arrangements

The source code of the recursive procedure **cpos** is presented in an easy to read basic pseudo code. All variables are integers and all are public except of `x`, `y` and **mBc**. The main program askes for the values of *m*, *n* and *D*, calculates *B*, initializes the array `sq()` as shown above and calls the procedure by **cpos(1,1)**. Bad domino constellations were count in **Bc** and compared with *B*. At the end the value of the variable **Scnt** is the number of different packings. Source code of the recursive procedure **cpos**`(x,y)` are presented in **Appendix**.
Different cases for the planned domino as they occur in the procedure are shown in Figure 25.

The most important results obtained with a computer program are summarized in four tables.

Table 1. Number of domin packings in $[n \times n]$-squares with $d(n) = d_0(n)$ (including reflections and rotations)

n	$D_0(n)$	P(n)
2	2	2
3	3	4
4	6	100
5	9	312
6	12	14
7	17	5020
8	22	4804
9	27	16
10	34	14844
11	41	11128
12	48	16
13	57	7568
14	66	4900
15	75	16
16	86	964
17	97	560

n	$D_0(n)$	P(n)
18	108	16
19	121	0
20	134	16
21	147	16
22	162	0
23	177	0
24	192	16
25	209	0
26	226	0
27	243	16
28	262	0
29	281	0
30	300	16
31	321	0
32	342	0
33	363	16

Table 2. Smallest number $D(m, n)$ of dominoes for which a packing exists

m \ n	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
14	66	70	76	80	84	90	94	98	104	108	112	118	122	126	132	136	140	146	150	154
15		75	80	85	90	95	100	105	110	115	120	125	130	135	140	145	150	155	160	165
16			86	92	96	102	108	112	118	124	128	134	140	144	150	156	160	166	172	176
17				97	102	109	114	119	126	131	136	143	148	153	160	165	170	177	182	187
18					108	114	120	126	132	138	144	150	156	162	168	174	180	186	192	198
19						122	128	133	141	147	152	160	166	171	179	185	190	198	204	209
20							134	140	148	154	160	168	174	180	188	194	200	208	214	220
21								147	154	161	168	175	182	189	196	203	210	217	224	231
22									163	170	176	185	192	198	207	214	220	229	236	242
23										178	184	193	201	207	216	224	230	239	247	253
24											192	200	208	216	224	232	240	248	256	264
25												210	218	225	235	243	250	260	268	275
26													227	234	244	253	260	270	279	286
27														243	252	261	270	279	288	297
28															263	272	280	291	300	308
29																282	290	301	311	319
30																	300	310	320	330
31																		322	332	341
32																			343	352
33																				363

In the coloured cases: $D(m, n) = d_0(m, n) + 1$

Table 3. Number of minimal domino packings in $[m \times n]$-rectangles with $D_0(m, n)$ dominoes (including reflections and rotations)

m \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	2	1	3	3	1	6	4	1	10	5	1	15	6	1	21	7	1	28
2		2	2	6	16	2	12	48	2	18	98	2	24	166	2	30	252	2	36	356
3			4	7	6	8	14	12	14	23	20	22	34	30	32	47	42	44	62	56
4				100	51	10	663	124	8	1892	178	8	3652	224	8	5796	274	8	8380	324
5					312	8	167	1394	10	327	3704	12	531	7376	14	779	12600	16	1071	19616
6						14	10	10	16	10	10	16	10	10	16	10	10	16	10	10
7							5020	268	8	12503	325	8	21688	430	8	34371	561	8	51242	710
8								4804	8	168	8546	8	134	10296	8	148	12704	8	166	16294
9									16	8	8	16	8	8	16	8	8	16	8	8
10										14844	86	8	14184	84	8	15282	98	8	18102	112
11											11128	8	20	10286	8	16	12352	8	16	16760
12												16	8	8	16	8	8	16	8	8
13													7568	4	8	4930	4	8	5064	4
14														4900	8	0	3456	8	0	4100
15															16	8	8	16	8	8
16																964	0	8	482	0
17																	560	8	0	328
18																		16	8	8
19																			0	0
20																				16

m \ n	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	8	1	36	9	1	45	10	1	55	11	1	66	12	1	78	13	1	91	14	1
2	2	42	478	2	48	618	2	54	776	2	60	952	2	66	1146	2	72	1358	2	78
3	58	79	72	74	98	90	92	119	110	112	142	132	134	167	156	158	194	182	184	223
4	8	11424	374	8	14920	424	8	18868	474	8	23268	524	8	28120	574	8	33424	624	8	39180
5	18	1407	28664	20	1787	39984	22	2211	53816	24	2679	70400	26	3191	89976	28	3747	112784	30	4347
6	16	10	10	16	10	10	16	10	10	16	10	10	16	10	10	16	10	10	16	10
7	8	72687	877	8	99148	1062	8	131129	1265	8	169134	1486	8	213667	1725	8	265232	1982	8	324333
8	8	184	20506	8	202	25204	8	220	30380	8	238	36034	8	256	42166	8	274	48776	8	292
9	16	8	8	16	8	8	16	8	8	16	8	8	16	8	8	16	8	8	16	8
10	8	21566	126	8	25286	140	8	29262	154	8	33494	168	8	37982	182	8	42726	196	8	47726
11	8	16	22400	8	16	29204	8	16	37272	8	16	46712	8	16	57632	8	16	70140	8	16
12	16	8	8	16	8	8	16	8	8	16	8	8	16	8	8	16	8	8	16	8
13	8	5680	4	8	6296	4	8	6912	4	8	7528	4	8	8144	4	8	8760	4	8	9376
14	8	0	5078	8	0	6174	8	0	7380	8	0	8696	8	0	10122	8	0	11658	8	0
15	16	8	8	16	8	8	16	8	8	16	8	8	16	8	8	16	8	8	16	8
16	8	482	0	8	482	0	8	482	0	8	482	0	8	482	0	8	482	0	8	482
17	8	0	360	8	0	400	8	0	440	8	0	480	8	0	520	8	0	560	8	0
18	16	8	8	16	8	8	16	8	8	16	8	8	16	8	8	16	8	8	16	8
19	8	0	0	8	0	0	8	0	0	8	0	0	8	0	0	8	0	0	8	0
20	8	0	8	8	0	8	8	0	8	8	0	8	8	0	8	8	0	8	8	0
21	16	8	8	16	8	8	16	8	8	16	8	8	16	8	8	16	8	8	16	8
22		0	0	8	0	0	8	0	0	8	0	0	8	0	0	8	0	0	8	0
23			0	8	0	0	8	0	0	8	0	0	8	0	0	8	0	0	8	0
24				16	8	8	16	8	8	16	8	8	16	8	8	16	8	8	16	8
25					0	0	8	0	0	8	0	0	8	0	0	8	0	0	8	0
26						0	8	0	0	8	0	0	8	0	0	8	0	0	8	0
27							16	8	8	16	8	8	16	8	8	16	8	8	16	8
28								0	0	8	0	0	8	0	0	8	0	0	8	0
29									0	8	0	0	8	0	0	8	0	0	8	0
30										16	8	8	16	8	8	16	8	8	16	8
31											0	0	8	0	0	8	0	0	8	0
32												0	8	0	0	8	0	0	8	0
33													16	8	8	16	8	8	16	8
34														0	0	8	0	0	8	0
35															0	8	0	0	8	0
36																16	8	8	16	8
37																	0	0	8	0
38																		0	8	0
39																			16	8
40																				0

In the coloured cases it was proved by exhaustive computer search that packings with $D_0(m, n)$ dominoes do not exist.

4. Some generalizations

A natural generalization is the cubic lattice. An estimate

$$D(m,n,k) \geq \left\lceil \frac{mnk}{3} \right\rceil \tag{17}$$

can also be used for a three-dimensional rectangles. As in two dimensions, the estimate (17) is sharp if any of the edge lengths is a multiple of 3. In this case, the minimum packing is obtained by repeating the minimum two-dimensional rectangles in layers. For illustration see Figure 26 with the minimal packing of $[3\times4\times5]$. More complex packings are shown in Figure 27.

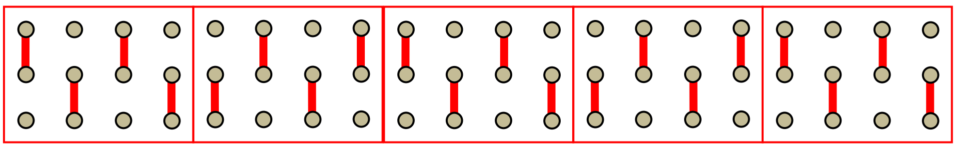

Figure 26. Minimal packing of $[3\times4\times5]$ in layers

The results obtained by using a computer program are summarized in Table 4.

Table 4. Number of minimal packings of $[n \times n \times n]$ cube

***D*(*n*, *n*, *n*)**	**Number of packings**
$D(1, 1, 1) = 0$	1
$D(2, 2, 2) = 3$	8
$D(3, 3, 3) = 9$	6
$D(4, 4, 4) = 22$	912
$D(5, 5, 5) = 43$	52 608
$D(6, 6, 6) = 72$	6

From Table 4 we see that $[5 \times 5 \times 5]$ cube is the smallest one for which estimate (17) is no longer sharp. The result $D(4, 4, 4) = 22$ is significant in that the estimate (17) is sharp, but the minimal packing cannot be obtained from the minimal packing of $[4 \times 4]$ rectangles.

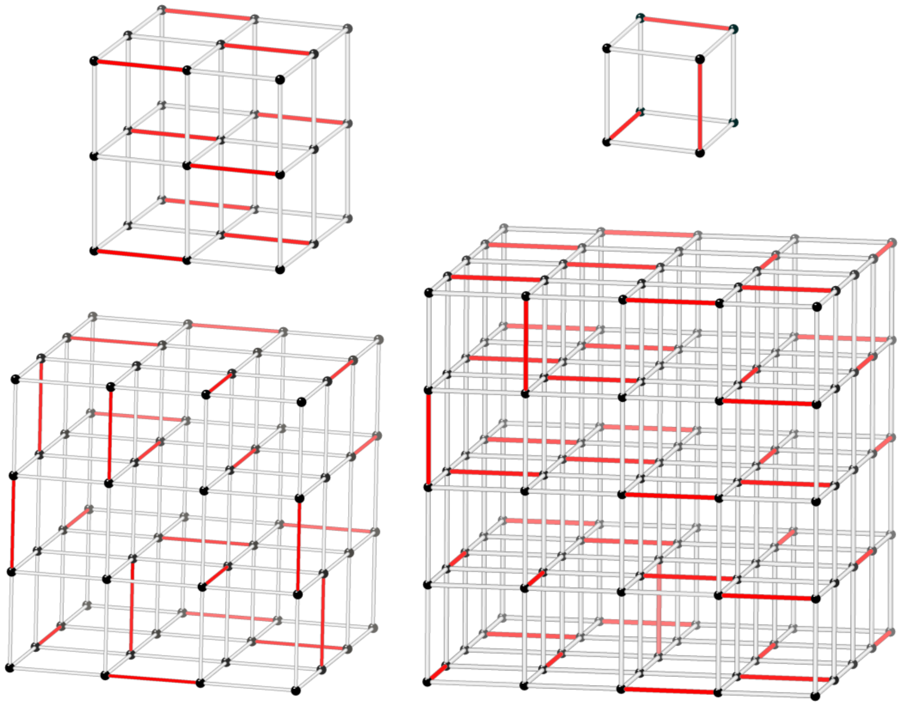

Figure 27. Minimal packings of cubes for $n = 2, 3, 4, 5$

5. Conclusions

The article contains the theorems of pure mathematics, as well as computer-assisted proofs. New progress has been made in solving the domino exclusion problem, including a deeper understanding of the structure of minimal packings. Proof (or disproof) of the hypothesis formulated in the Remark 4 could be a natural continuation of this study.

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Appendix: Source code of the recursive procedure **cpos(x,y)**

```
Proc cPos(x, y)
  Switch sq(x, y) look at the current cell
  Case 0   the current cell is empty
    ---------------------leave the cell empty if possible
    If sq(x - 1, y)          no hole on the left?
      If sq(x, y - 1)          no hole above?
        cPos(x + 1, y)
    ---------------------------- memorize Bc
    Local Int mBc = Bc
    --------------------try to put a horizontal domino
    If sq(x + 1, y) = 0  is the right neighbor cell empty?
      Switch sq(x - 1, y)  left neighbor cell of the domino
        Case -1    : Bc += 2
        Case  2    : Bc++
        Case -2, 5 : Bc++ : If sq(x, y - 1) Then Bc++
        Case  3, 4 :        If sq(1, y - 1) Then Bc++
      Switch sq(x + 2, y) right neighbor cell of the domino
        Case -2, -5 : Bc++ : If sq(x + 1, y - 1) Then Bc++
        Case -3, -4 :        If sq(n, y - 1)     Then Bc++
      If Bc <= B
        sq(x, y) = 1 : sq(x + 1, y) = -1  put horizontal domino
        cPos(x + 2, y)
        sq(x, y) = 0 : sq(x + 1, y) = 0   delete domino
      Bc = mBc
    ------------------------ try to put a vertical domino
    If sq(x, y + 1) = 0    is the cell below empty?
      Switch sq(x - 1, y)  left neighbor cell of the domino
        Case -1    : Bc++
        Case  2    : Bc += 2
        Case -2, 3 :        If sq(x, y - 1) Then Bc++
        Case  4    : Bc++ : If sq(x, y - 1) Then Bc++
      Switch sq(x + 1, y) right neighbor cell of the domino
        Case -2, -3 :        If sq(x, y - 1) Then Bc++
        Case -4     : Bc++ : If sq(x, y - 1) Then Bc++
      If Bc <= B
        sq(x, y) = 2 : sq(x, y + 1) = -2  put vertical domino
        cPos(x + 1, y)
        sq(x, y) = 0 : sq(x, y + 1) = 0   delete domino
      Bc = mBc
  Case -2          the current cell is already occupied
    cPos(x + 1, y)
  Case -3, -4      the end of a row is reached
    cPos(1, y + 1)
  Case -5          the end of the last row is reached
    If Bc = B Then Scnt++
```